REPEATED EIGENVALUES

$$\vec{X}' = \begin{bmatrix} 1 & -1 \\ 1 & 3 \end{bmatrix} \vec{X} \qquad A = \begin{bmatrix} 1 & -1 \\ 1 & 3 \end{bmatrix}$$

Let $\vec{X} = \vec{\xi} e^{rt}$

$$\Rightarrow \vec{X}' = \vec{\xi} r e^{rt}$$

Substituting, we get $\underbrace{\vec{\xi} r e^{rt}}_{\vec{X}'} = A \underbrace{\xi e^{rt}}_{\vec{X}}$

$$\Rightarrow (A - rI)\vec{\xi} = 0$$

Eigenvalues: $|A - rI| = 0 \Rightarrow \begin{vmatrix} 1-r & -1 \\ 1 & 3-r \end{vmatrix} = 0$

$$\Rightarrow (1-r)(3-r) + 1 = 0$$

$$\rightarrow r^2 - 4r + 4 = 0 \rightarrow (r-2)^2 = 0$$

$$\therefore r_1 = r_2 = 2$$

Eigenvector: $(A - r_1 I)\xi = 0 \Rightarrow \begin{bmatrix} -1 & -1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} \xi_1 \\ \xi_2 \end{bmatrix} = 0$

$$\Rightarrow \left.\begin{aligned} -\xi_1 - \xi_2 &= 0 \\ \xi_1 + \xi_2 &= 0 \end{aligned}\right\} \vec{\xi} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

Since we have only one eigenvalue, we get only one eigenvector. We need two solution to get the general solution.

First solution, $\vec{X}^{(1)} = \vec{\xi}^{(1)} e^{r_1 t} = \begin{bmatrix} 1 \\ -1 \end{bmatrix} e^{2t}$

Recall that in second order equations, when we had repeated roots, if first solution was $y_1 = e^{rt}$, then the second solution is

$$y_2 = te^{rt}.$$

Using the same logic, we first try

$$\vec{x} = \vec{\xi} t e^{rt}$$

Therefore, $\vec{x}' = \vec{\xi} t r e^{rt} + \vec{\xi} e^{rt}$

Substituting in $\vec{x}' = A\vec{x}$, we get

$$\vec{\xi} t r e^{rt} + \vec{\xi} e^{rt} = A \vec{\xi} t e^{rt}$$

Combining similar terms, we have

$$(\vec{\xi} r - A\vec{\xi}) t e^{rt} + \vec{\xi} e^{rt} = 0$$

$$\Rightarrow \quad -(A - rI)\vec{\xi} t e^{rt} + \vec{\xi} e^{rt} = 0$$

Equating coefficients of te^{rt} & e^{rt} to zero, we have

$$\underbrace{(A - rI)\vec{\xi} = 0}_{\text{Same as before.}} \quad \text{and} \quad \vec{\xi} = 0$$

$\downarrow$ Gives us a zero solution.

We get $r = 2$, $\vec{\xi} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$

Therefore $\vec{x}^{(2)} \neq \vec{\xi} t e^{rt}$.

To get a non-zero solution, we need to include a new term $\vec{\eta}e^{rt}$, i.e;

$$\vec{X} = \vec{\xi}te^{rt} + \vec{\eta}e^{rt}, \quad \text{with} \quad r=2.$$

$$\therefore \vec{X}' = \vec{\xi}e^{rt} + \vec{\xi}tre^{rt} + \vec{\eta}re^{rt}.$$

Substituting, we get

$$\vec{\xi}e^{rt} + \vec{\xi}tre^{rt} + \vec{\eta}re^{rt} = A(\vec{\xi}te^{rt} + \vec{\eta}e^{rt})$$

Equating similar terms on both sides:

$$\underline{O(te^{rt})}: \quad r\vec{\xi} = A\vec{\xi} \Rightarrow (A-rI)\vec{\xi} = 0 \quad \text{— ①}$$

$$\underline{O(e^{rt})}: \quad \vec{\xi} + r\vec{\eta} = A\vec{\eta} \Rightarrow (A-rI)\vec{\eta} = \vec{\xi} \quad \text{— ②}$$

Equation ① is same as before: We get $r=2$, $\vec{\xi} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$

We now find $\vec{\eta}$ from equation ②:

$$(A-2I)\vec{\eta} = \vec{\xi}$$

$$\Rightarrow \begin{bmatrix} -1 & -1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} \eta_1 \\ \eta_2 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \end{bmatrix} \Rightarrow \left.\begin{array}{l} -\eta_1 - \eta_2 = 1 \\ \& \ \eta_1 + \eta_2 = -1 \end{array}\right\} \text{same equation.}$$

$$\therefore \eta_1 + \eta_2 = -1.$$

If $\eta_1 = K$, $\eta_2 = -1-K$

$$\therefore \vec{\eta} = \begin{bmatrix} K \\ -1-K \end{bmatrix} = \begin{bmatrix} 0 \\ -1 \end{bmatrix} + K\begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

$\therefore \vec{X}^{(2)} = \vec{\xi}\, t e^{rt} + \vec{\eta}\, e^{rt}$

$= \begin{bmatrix} 1 \\ -1 \end{bmatrix} t e^{2t} + \left(\begin{bmatrix} 0 \\ -1 \end{bmatrix} + k \begin{bmatrix} 1 \\ -1 \end{bmatrix} \right) e^{2t}$

↳ Same as the first solution, $\vec{X}^{(1)}$!

$\therefore \vec{X}^{(2)} = \begin{bmatrix} 1 \\ -1 \end{bmatrix} t e^{2t} + \begin{bmatrix} 0 \\ -1 \end{bmatrix} e^{2t}$.

Therefore, general solution is

$\vec{X} = c_1 \vec{X}^{(1)} + c_2 \vec{X}^{(2)}$

$= c_1 \begin{bmatrix} 1 \\ -1 \end{bmatrix} e^{2t} + c_2 \left[\begin{bmatrix} 1 \\ -1 \end{bmatrix} t e^{2t} + \begin{bmatrix} 0 \\ -1 \end{bmatrix} e^{2t} \right]$

__Summary:__ (Repeated roots) $\vec{X}' = A\vec{X}$

① Put $\vec{X} = \vec{\xi} e^{rt}$ → Obtain $(A - rI)\vec{\xi} = 0$ ↓ Find r & $\vec{\xi}$

First solution : $\vec{X}^{(1)} = \vec{\xi} e^{rt}$

② Second solution : Put $\vec{X}^{(2)} = \vec{\xi} t e^{rt} + \vec{\eta} e^{rt}$ with a known r. Substitute $\vec{X}^{(2)}$ into $\vec{X}' = A\vec{X}$ to obtain : $(A - rI)\vec{\eta} = \vec{\xi}$.

③ Find $\vec{\eta}$. Extract the part involving $\vec{\xi}$ & discard it.

$\therefore \quad \vec{X}^{(2)} = \vec{\xi} t e^{rt} + \vec{\eta} e^{rt}$

4 General Solution: $\vec{X} = c_1 \vec{X}^{(1)} + c_2 \vec{X}^{(2)}$

Ex: $\vec{X}' = A\vec{X}$ where $A = \begin{bmatrix} 3 & -4 \\ 1 & -1 \end{bmatrix}$

Step 1: Put $\vec{X} = \vec{\xi} e^{rt}$

$\Rightarrow \vec{\xi} r e^{rt} = A\vec{\xi} e^{rt} \rightarrow (A - rI)\vec{\xi} = 0$

Eigenvalues: $|A - rI| = 0 \rightarrow (3-r)(-1-r) + 4 = 0$

$\rightarrow r^2 - 2r + 1 = 0$

$\rightarrow (r-1)^2 = 0 \Rightarrow r_1 = r_2 = 1$

Eigenvectors: $(A - rI)\vec{\xi} = 0 \rightarrow \begin{bmatrix} 2 & -4 \\ 1 & -2 \end{bmatrix}\begin{bmatrix} \xi_1 \\ \xi_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$\Rightarrow \left.\begin{array}{l} 2\xi_1 - 4\xi_2 = 0 \\ \xi_1 - 2\xi_2 = 0 \end{array}\right\} \quad \xi_1 = 2\xi_2$

Let $\xi_2 = 1$, then $\xi_1 = 2 \Rightarrow \vec{\xi} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$

$\therefore \quad \vec{X}^{(1)} = \begin{bmatrix} 2 \\ 1 \end{bmatrix} e^t$

Step 2: Second Solution: Let $\vec{X} = \vec{\xi} t e^{rt} + \vec{\eta} e^{rt}$

We get $(A - rI)\vec{\eta} = \vec{\xi}$

Step 3: Since $r = 1$ & $\vec{\xi} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$, we have

$$\begin{bmatrix} 2 & -4 \\ 1 & -2 \end{bmatrix} \begin{bmatrix} \eta_1 \\ \eta_2 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

$$\longrightarrow \left.\begin{array}{l} 2\eta_1 - 4\eta_2 = 2 \\ \eta_1 - 2\eta_2 = 1 \end{array}\right\} \quad \eta_1 = 1 + 2\eta_2$$

Let $\eta_2 = K \Rightarrow \eta_1 = 1 + 2K$

$$\therefore \vec{\eta} = \begin{bmatrix} 1+2K \\ K \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} + k \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

↳ Same as $\vec{\xi}$.

$\Rightarrow$ Discard it.

$$\therefore \vec{X}^{(2)} = \vec{\xi}\, t e^{rt} + \vec{\eta} e^{rt}$$

$$= \begin{bmatrix} 2 \\ 1 \end{bmatrix} t e^{t} + \begin{bmatrix} 1 \\ 0 \end{bmatrix} e^{t}$$

Step 4: General solution:

$$\vec{X} = c_1 \vec{X}^{(1)} + c_2 \vec{X}^{(2)}$$

$$= c_1 \begin{bmatrix} 2 \\ 1 \end{bmatrix} e^{t} + c_2 \left\{ \begin{bmatrix} 2 \\ 1 \end{bmatrix} t e^{t} + \begin{bmatrix} 1 \\ 0 \end{bmatrix} e^{t} \right\}$$