Section 7.5 Homogeneous Linear Systems with Constant Coefficients.

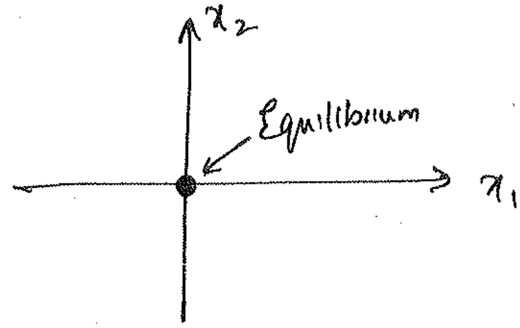
$$\vec{x}' = A\vec{x}$$

This is a linear differential equation for a vector.

Equilibrium Solution: $A\vec{x} = 0 \Rightarrow \vec{x} = 0$

(Recall for $\frac{dy}{dt} = f(y)$, equilibrium was $f(y) = 0$)

Let $\vec{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \Rightarrow$ Equilibrium is $x_1 = 0$ & $x_2 = 0$



Now we want to examine (ie; plot) the solutions of the linear homogeneous equation $\vec{x}' = A\vec{x}$. We keep A as a constant matrix. Recall that a solution of the first order linear differential equation

$\frac{dy}{dt} = ay$ can be obtained using the

Substitution $y = e^{rt} \Rightarrow y' = re^{rt}$. Substituting, we get

$re^{rt} = ae^{rt} \Rightarrow r = a. \Rightarrow$ Solution of $\frac{dy}{dt} = ay$ is $y = ce^{at}$

We now use the same logic for $\vec{x}' = A\vec{x}$.

Let $\vec{x} = \vec{\xi} e^{rt}$

has magnitude & direction

Controls the direction

Controls the magnitude.

Note that $\vec{\xi}$ is purely a spatial component.

Therefore $\vec{x}' = \frac{d\vec{x}}{dt} = \vec{\xi} r e^{rt}$

Substituting $\vec{x}'$ in to $\vec{x} = A\vec{x}$, we get

$$\vec{\xi} r e^{rt} = A\vec{\xi} e^{rt}$$

$$\Rightarrow (A\vec{\xi} - r\vec{\xi}) e^{rt} = 0$$

Since $e^{rt} \neq 0$, we have $(A - rI)\vec{\xi} = 0$

where $I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$: Identity matrix.

This is the Eigenvalue - Eigenvector problem for the matrix A.

$r \longrightarrow$ Eigenvalue

$\vec{\xi} \longrightarrow$ Eigenvector.

Ex: $\vec{x}' = \underbrace{\begin{bmatrix} 1 & 1 \\ 4 & 1 \end{bmatrix}}_{A} \vec{x}$

Let $\vec{x} = \vec{\xi} e^{rt}$

$\Rightarrow \vec{x}' = \vec{\xi} r e^{rt}$

$\Rightarrow \underbrace{\vec{\xi} r e^{rt}}_{\vec{x}'} = A \underbrace{\vec{\xi} e^{rt}}_{\vec{x}}$

$\Rightarrow (A\vec{\xi} - r\vec{\xi}) e^{rt} = 0$

$\therefore (A - rI)\vec{\xi} = 0$

Eigenvalues: finding r :

$|A - rI| = 0 \Rightarrow \left| \begin{bmatrix} 1 & 1 \\ 4 & 1 \end{bmatrix} - r \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right| = 0$

$\Rightarrow \begin{vmatrix} 1-r & 1 \\ 4 & 1-r \end{vmatrix} = 0 \quad \Rightarrow (1-r)^2 - 4 = 0$

$\Rightarrow r^2 - 2r - 3 = 0 \quad \Rightarrow (r-3)(r+1) = 0$

$\therefore r_1 = 3 \quad \& \quad r_2 = -1$

Eigenvectors: with $r_1 = 3$:

Using $(A - rI)\vec{\xi} = 0$, we have

$$\begin{bmatrix} -2 & 1 \\ 4 & -2 \end{bmatrix} \begin{bmatrix} \xi_1 \\ \xi_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \qquad \text{where} \quad \vec{\xi} = \begin{bmatrix} \xi_1 \\ \xi_2 \end{bmatrix}$$

$$\Rightarrow \left.\begin{array}{l} -2\xi_1 + \cancel{2}\,\xi_2 = 0 \\ \text{and} \quad 4\xi_1 - 2\xi_2 = 0 \end{array}\right\} \quad \xi_2 = 2\xi_1$$

Notice that both the equation resulted in only one relation between ξ_1 & ξ_2.

Choosing $\xi_1 = 1$, we get $\xi_2 = 2$

$$\Rightarrow \quad \vec{\xi}^{(1)} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}.$$

This is not the only solution. In fact eigenvectors are unique only upto a multiplicative constant.

Superscript $^{(1)}$ indicates that is is the first eigenvector.

Second eigenvector: with $r_2 = -1$:

$$\underbrace{\begin{bmatrix} 2 & 1 \\ 4 & 2 \end{bmatrix} \begin{bmatrix} \xi_1 \\ \xi_2 \end{bmatrix}}_{} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$\hookrightarrow \vec{\xi}^{(2)}$

$$\Rightarrow \quad \left.\begin{array}{l} 2\xi_1 + \xi_2 = 0 \\ 4\xi_1 + 2\xi_2 = 0 \end{array}\right\} \; -2\xi_1 = \xi_2$$

Choosing $\xi_1 = 1$, we have $\xi_2 = -2$

$$\Rightarrow \quad \vec{\xi}^{(2)} = \begin{bmatrix} 1 \\ -2 \end{bmatrix}$$

Since $\vec{X} = \vec{\xi} e^{rt}$, we have two solutions :

$$\vec{X}^{(1)} = \vec{\xi}^{(1)} e^{r_1 t} = \begin{bmatrix} 1 \\ 2 \end{bmatrix} e^{3t}$$

and $$\vec{X}^{(2)} = \vec{\xi}^{(2)} e^{r_2 t} = \begin{bmatrix} 1 \\ -2 \end{bmatrix} e^{-t}$$

Before we write the general solution, we should make sure $\vec{X}^{(1)}$ & $\vec{X}^{(2)}$ indeed form a fundamental set. This can be verified by making sure Wronskian is non-zero.

$$W[\vec{X}^{(1)} \; \vec{X}^{(2)}] = \begin{vmatrix} X_1^{(1)} & X_1^{(2)} \\ X_2^{(1)} & X_2^{(2)} \end{vmatrix}$$

$$= \begin{vmatrix} e^{3t} & e^{-t} \\ 2e^{3t} & -2e^{-t} \end{vmatrix} = -4e^{2t} \neq 0$$

<u>General</u> <u>Solution</u> :- $\vec{X} = c_1 \vec{X}^{(1)} + c_2 \vec{X}^{(2)}$ where c_1 & c_2 are arbitrary constants.

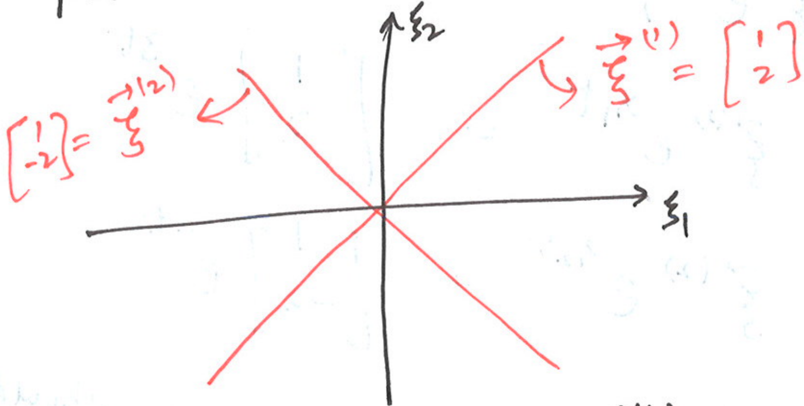
$$\Rightarrow \vec{X} = c_1 \begin{bmatrix} 1 \\ 2 \end{bmatrix} e^{3t} + c_2 \begin{bmatrix} 1 \\ -2 \end{bmatrix} e^{-t}$$

Notice that $\vec{X}^{(1)} = \begin{bmatrix} 1 \\ 2 \end{bmatrix} e^{3t} \to \infty$ as $t \to \infty$, and

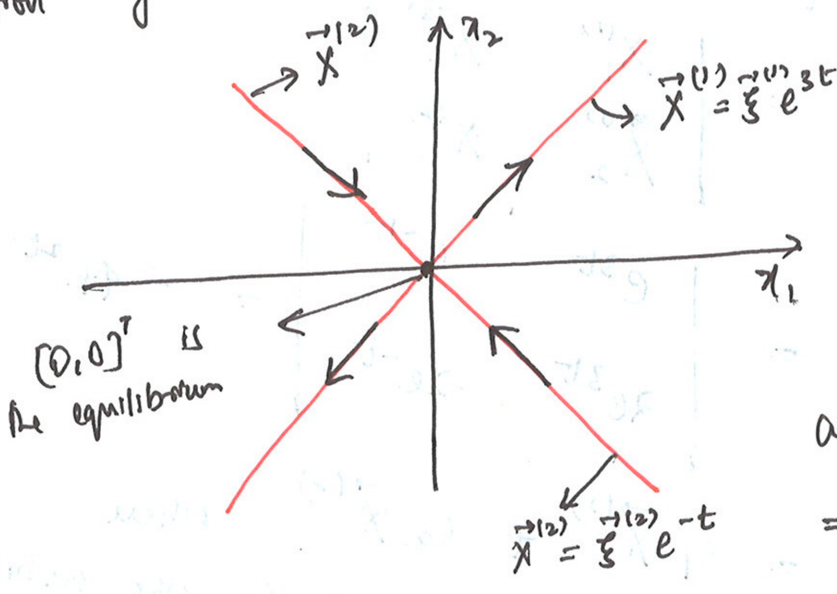
$$\vec{X}^{(2)} = \begin{bmatrix} 1 \\ -2 \end{bmatrix} e^{-t} \to \vec{0} \text{ as } t \to \infty$$

Plotting procedure :-

① First plot the two eigenvectors, $\vec{\xi}^{(1)}$ & $\vec{\xi}^{(2)}$



② Now $\vec{X}^{(1)}$ & $\vec{X}^{(2)}$ lie on $\vec{\xi}^{(1)}$ & $\vec{\xi}^{(2)}$ but change with time. To indicate time, use arrows, ie; the direction of the arrow shows increasing time.



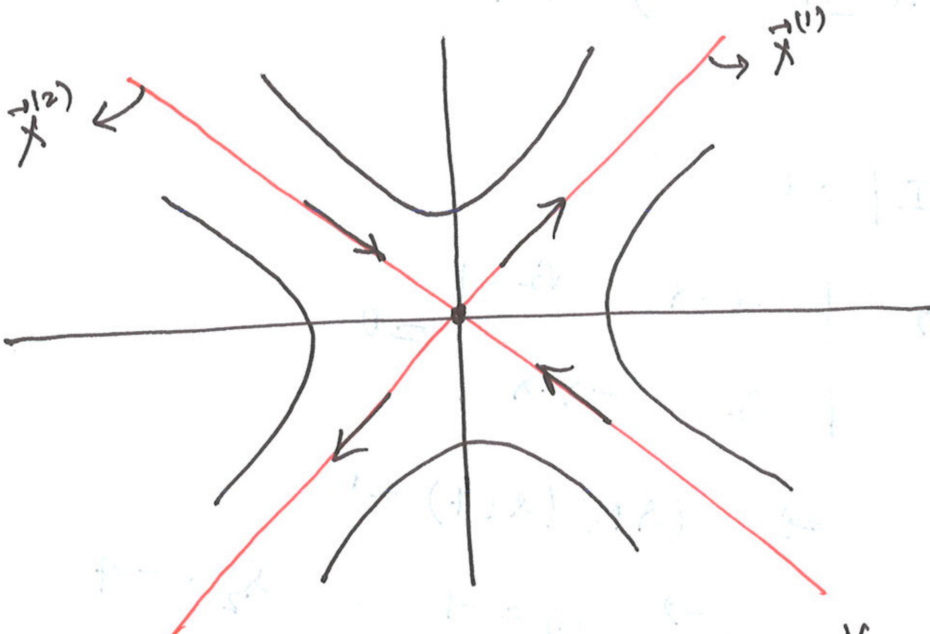
Recall, $\vec{X}^{(1)} \to \infty$ as $t \to \infty$

$\Rightarrow$ You move away from the equilibrium

and $\vec{X}^{(2)} \to 0$ as $t \to \infty$

$\Rightarrow$ You move ~~to~~toward equilibrium

3 The general solution is obtained for arbitrary values of c_1 & c_2. Therefore the general solution is represented with curves consistent with the arrows that you have already drawn.



The equilibrium in this case is a saddle point.

Notice that you never reach the equilibrium point unless you start exactly on $\vec{X}^{(2)}$. Even in this case, you have to wait for an infinite amount of time to reach $\begin{bmatrix}0\\0\end{bmatrix}$ since

$$\vec{X}^{(2)} \longrightarrow 0 \quad \text{as} \quad t \to \infty.$$

What if the eigenvalues have the same sign?

Ex: $\vec{X}' = \underbrace{\begin{bmatrix} -3 & \sqrt{2} \\ \sqrt{2} & -2 \end{bmatrix}}_{A} \vec{X}$

Using $\vec{X} = \vec{\xi} e^{rt}$, we gain get

$$(A\vec{\xi} - r\vec{\xi}) e^{rt} = 0 \quad \Rightarrow \quad (A - rI)\vec{\xi} = 0$$

Eigenvalues: $|A - rI| = 0$

$$\Rightarrow \begin{vmatrix} -3-r & \sqrt{2} \\ \sqrt{2} & -2-r \end{vmatrix} = 0$$

$$\Rightarrow (r+1)(r+4) = 0$$

$$\Rightarrow r_1 = -1 \ \& \ r_2 = -4$$

same sign.

Eigenvectors:- with $r_1 = -1$, we get from $(A - rI)\vec{\xi} = 0$,

$$\vec{\xi}^{(1)} = \begin{bmatrix} 1 \\ \sqrt{2} \end{bmatrix}$$

with $r_2 = -4$, we get $\vec{\xi}^{(2)} = \begin{bmatrix} -\sqrt{2} \\ 1 \end{bmatrix}$

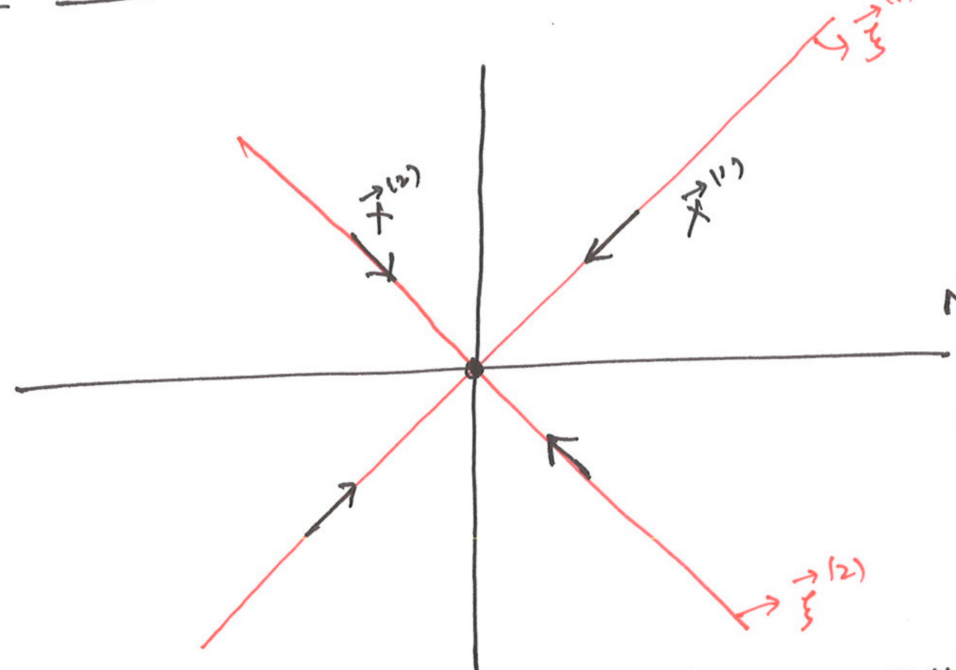
$$\therefore \vec{X}^{(1)} = \begin{bmatrix} 1 \\ \sqrt{2} \end{bmatrix} e^{-t} \ ; \quad \vec{X}^{(2)} = \begin{bmatrix} -\sqrt{2} \\ 1 \end{bmatrix} e^{-4t}$$

Now as $t \to \infty$; $\vec{X}^{(1)} \to \vec{0}$ and $\vec{X}^{(2)} \to \vec{0}$.

But since e^{-t} goes to zero much more slowly than e^{-4t}, as $t \to \infty$, $\vec{X}^{(2)}$ becomes negligible.

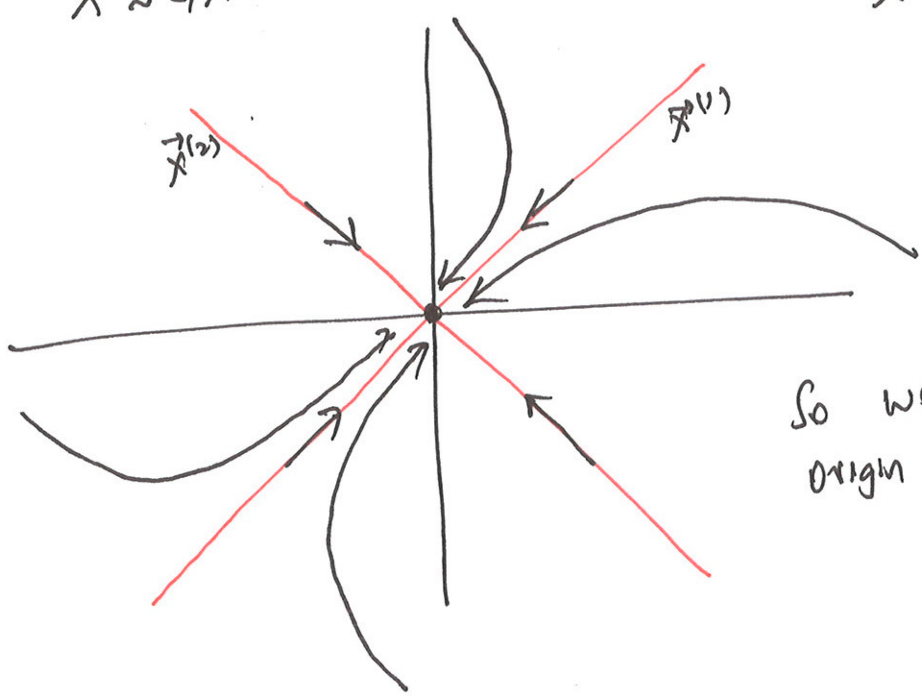
$\Rightarrow$ As $t \to \infty$; $\vec{X} \approx \vec{X}^{(1)}$

<u>General</u> <u>solution</u> $\vec{X} = c_1 \vec{X}^{(1)} + c_2 \vec{X}^{(2)}$



Fundamental Solutions.
Notice that all vectors are pointing towards origin.

As $t \to \infty$; $\vec{X} \approx c_1 \vec{X}^{(1)}$ since $\vec{X}^{(2)}$ goes to zero much faster than $\vec{X}^{(1)}$



So we approach the origin along $\vec{X}^{(1)}$