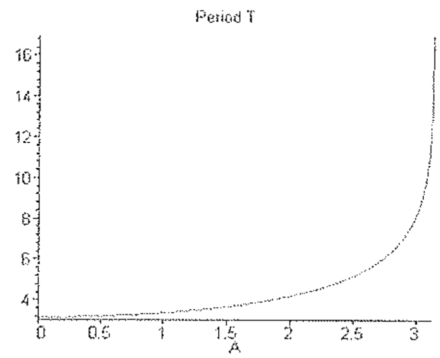
SECTION 9.4 # 1, 2, 4

9.5 # 1, 2

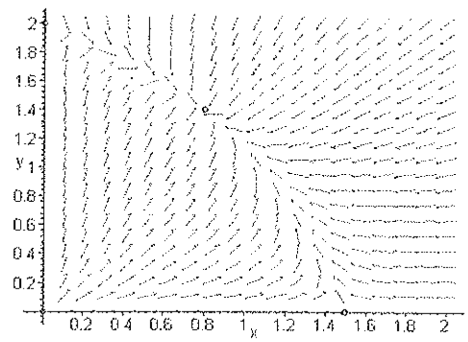
28. The characteristic equation of the coefficient matrix is $(r+1)^2 = 0$, with roots $r_1 = r_2 = -1$. Hence the critical point is an asymptotically stable node. On the other hand, the characteristic equation of the perturbed system is $r^2 + 2r + 1 + \epsilon = 0$, with roots $r_{1,2} = -1 \pm \sqrt{-\epsilon}$. If $\epsilon > 0$, then $r_{1,2} = -1 \pm i\sqrt{\epsilon}$ are complex roots. The critical point is a stable spiral. If $\epsilon < 0$, then $r_{1,2} = -1 \pm \sqrt{|\epsilon|}$ are real and both negative ($|\epsilon| \ll 1$). The critical point remains a stable node.

29.(d) Set $k = \sin(\alpha/2) = \sin(A/2)$ and $g/L = 4$.



9.4

1.(a)



(b) The critical points are solutions of the system of equations

$$x(1.5 - x - 0.5\,y) = 0$$
$$y(2 - y - 0.75\,x) = 0\,.$$

The four critical points are $(0,0)$, $(0,2)$, $(1.5,0)$ and $(0.8,1.4)$.

(c) The Jacobian matrix of the vector field is

$$\mathbf{J} = \begin{pmatrix} 3/2 - 2x - y/2 & -x/2 \\ -3y/4 & 2 - 3x/4 - 2y \end{pmatrix}.$$

At the critical point $(0,0)$, the coefficient matrix of the linearized system is

$$\mathbf{J}(0,0)=\begin{pmatrix}3/2 & 0\\ 0 & 2\end{pmatrix}.$$

The eigenvalues and eigenvectors are

$$r_1=3/2\,,\ \boldsymbol{\xi}^{(1)}=\begin{pmatrix}1\\0\end{pmatrix}\ ;\ r_2=2\,,\ \boldsymbol{\xi}^{(2)}=\begin{pmatrix}0\\1\end{pmatrix}.$$

The eigenvalues are positive, hence the origin is an unstable node.
At the critical point $(0,2)$, the coefficient matrix of the linearized system is

$$\mathbf{J}(0,2)=\begin{pmatrix}1/2 & 0\\ -3/2 & -2\end{pmatrix}.$$

The eigenvalues and eigenvectors are

$$r_1=1/2\,,\ \boldsymbol{\xi}^{(1)}=\begin{pmatrix}1\\-0.6\end{pmatrix}\ ;\ r_2=-2\,,\ \boldsymbol{\xi}^{(2)}=\begin{pmatrix}0\\1\end{pmatrix}.$$

The eigenvalues are of opposite sign. Hence the critical point is a saddle, which is unstable.
At the critical point $(1.5,0)$, the coefficient matrix of the linearized system is

$$\mathbf{J}(1.5,0)=\begin{pmatrix}-1.5 & -0.75\\ 0 & 0.875\end{pmatrix}.$$

The eigenvalues and eigenvectors are

$$r_1=-1.5\,,\ \boldsymbol{\xi}^{(1)}=\begin{pmatrix}1\\0\end{pmatrix}\ ;\ r_2=0.875\,,\ \boldsymbol{\xi}^{(2)}=\begin{pmatrix}-0.31579\\1\end{pmatrix}.$$

The eigenvalues are of opposite sign. Hence the critical point is also a saddle, which is unstable.
At the critical point $(0.8,1.4)$, the coefficient matrix of the linearized system is

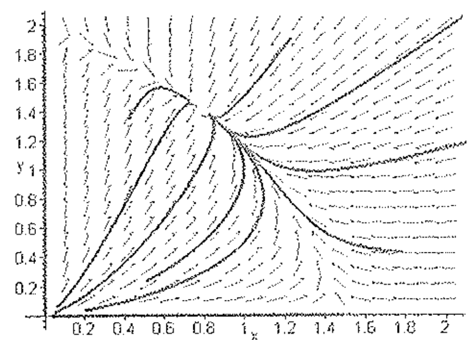
$$\mathbf{J}(0.8,1.4)=\begin{pmatrix}-0.8 & -0.4\\ -1.05 & -1.4\end{pmatrix}.$$

The eigenvalues and eigenvectors are

$$r_1=-\frac{11}{10}+\frac{\sqrt{51}}{10}\,,\ \boldsymbol{\xi}^{(1)}=\begin{pmatrix}1\\ \frac{3-\sqrt{51}}{4}\end{pmatrix}\ ;\ r_2=-\frac{11}{10}-\frac{\sqrt{51}}{10}\,,\ \boldsymbol{\xi}^{(2)}=\begin{pmatrix}1\\ \frac{3+\sqrt{51}}{4}\end{pmatrix}.$$

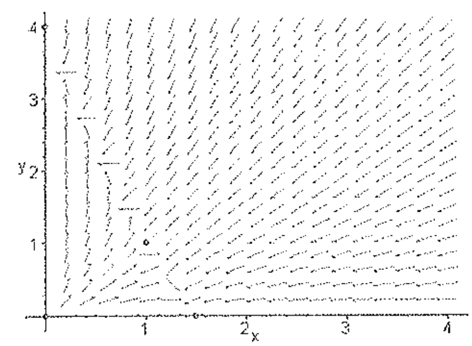
The eigenvalues are both negative. Hence the critical point is a stable node, which is asymptotically stable.

(d,e)



(f) Except for initial conditions lying on the coordinate axes, almost all trajectories converge to the stable node at $(0.8, 1.4)$.

2.(a)



(b) The critical points are the solution set of the system of equations

$$x(1.5 - x - 0.5\,y) = 0$$
$$y(2 - 0.5\,y - 1.5\,x) = 0\,.$$

The four critical points are $(0, 0)$, $(0, 4)$, $(1.5, 0)$ and $(1, 1)$.

(c) The Jacobian matrix of the vector field is

$$\mathbf{J} = \begin{pmatrix} 3/2 - 2x - y/2 & -x/2 \\ -3y/2 & 2 - 3x/2 - y \end{pmatrix}.$$

At the origin, the coefficient matrix of the linearized system is

$$\mathbf{J}(0, 0) = \begin{pmatrix} 3/2 & 0 \\ 0 & 2 \end{pmatrix}.$$

The eigenvalues and eigenvectors are

$$r_1 = 3/2\,,\ \boldsymbol{\xi}^{(1)} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}\,;\ r_2 = 2\,,\ \boldsymbol{\xi}^{(2)} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

The eigenvalues are positive, hence the origin is an unstable node.
At the critical point $(0\,,4)$, the coefficient matrix of the linearized system is

$$\mathbf{J}(0\,,4) = \begin{pmatrix} -1/2 & 0 \\ -6 & -2 \end{pmatrix}.$$

The eigenvalues and eigenvectors are

$$r_1 = -1/2\;,\; \boldsymbol{\xi}^{(1)} = \begin{pmatrix} 1 \\ -4 \end{pmatrix}\;;\; r_2 = -2\;,\; \boldsymbol{\xi}^{(2)} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

The eigenvalues are both negative, hence the critical point $(0\,,4)$ is a stable node, which is asymptotically stable.
At the critical point $(3/2\,,0)$, the coefficient matrix of the linearized system is

$$\mathbf{J}(3/2\,,0) = \begin{pmatrix} -3/2 & -3/4 \\ 0 & -1/4 \end{pmatrix}.$$

The eigenvalues and eigenvectors are

$$r_1 = -3/2\;,\; \boldsymbol{\xi}^{(1)} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}\;;\; r_2 = -1/4\;,\; \boldsymbol{\xi}^{(2)} = \begin{pmatrix} 3 \\ -5 \end{pmatrix}.$$

The eigenvalues are both negative, hence the critical point is a stable node, which is asymptotically stable.
At the critical point $(1, 1)$, the coefficient matrix of the linearized system is

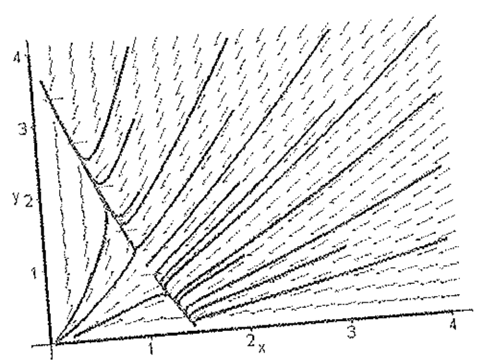
$$\mathbf{J}(1\,,1) = \begin{pmatrix} -1 & -1/2 \\ -3/2 & -1/2 \end{pmatrix}.$$

The eigenvalues and eigenvectors are

$$r_1 = \frac{-3+\sqrt{13}}{4}\;,\; \boldsymbol{\xi}^{(1)} = \begin{pmatrix} 1 \\ -\frac{1+\sqrt{13}}{2} \end{pmatrix}\;;\; r_2 = -\frac{3+\sqrt{13}}{4}\;,\; \boldsymbol{\xi}^{(2)} = \begin{pmatrix} 0 \\ \frac{-1+\sqrt{13}}{2} \end{pmatrix}.$$

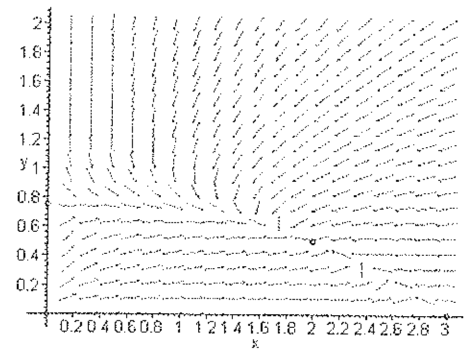
The eigenvalues are of opposite sign, hence $(1, 1)$ is a saddle, which is unstable.

(d,e)



(f) Trajectories approaching the critical point $(1, 1)$ form a separatrix. Solutions on either side of the separatrix approach either $(0, 4)$ or $(1.5, 0)$.

4.(a)



(b) The critical points are solutions of the system of equations

$$x(1.5 - 0.5\,x - y) = 0$$
$$y(0.75 - y - 0.125\,x) = 0\,.$$

The four critical points are $(0, 0)$, $(0, 3/4)$, $(3, 0)$ and $(2, 1/2)$.

(c) The Jacobian matrix of the vector field is

$$\mathbf{J} = \begin{pmatrix} 3/2 - x - y & -x \\ -y/8 & 3/4 - x/8 - 2y \end{pmatrix}.$$

At the origin, the coefficient matrix of the linearized system is

$$\mathbf{J}(0, 0) = \begin{pmatrix} 3/2 & 0 \\ 0 & 3/4 \end{pmatrix}.$$

The eigenvalues and eigenvectors are

$$r_1 = 3/2\,,\ \xi^{(1)} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}\,;\ r_2 = 3/4\,,\ \xi^{(2)} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

The eigenvalues are positive, hence the origin is an unstable node.
At the critical point $(0, 3/4)$, the coefficient matrix of the linearized system is

$$\mathbf{J}(0, 3/4) = \begin{pmatrix} 3/4 & 0 \\ -3/32 & -3/4 \end{pmatrix}.$$

The eigenvalues and eigenvectors are

$$r_1 = 3/4\,,\ \xi^{(1)} = \begin{pmatrix} -16 \\ 1 \end{pmatrix}\,;\ r_2 = -3/4\,,\ \xi^{(2)} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

The eigenvalues are of opposite sign, hence the critical point $(0, 3/4)$ is a saddle, which is unstable.

At the critical point $(3,0)$, the coefficient matrix of the linearized system is

$$\mathbf{J}(3,0) = \begin{pmatrix} -3/2 & -3 \\ 0 & 3/8 \end{pmatrix}.$$

The eigenvalues and eigenvectors are

$$r_1 = -3/2\,,\ \xi^{(1)} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}\ ;\ r_2 = 3/8\,,\ \xi^{(2)} = \begin{pmatrix} -8 \\ 5 \end{pmatrix}.$$

The eigenvalues are of opposite sign, hence the critical point $(0,3/4)$ is a saddle, which is unstable.
At the critical point $(2,1/2)$, the coefficient matrix of the linearized system is

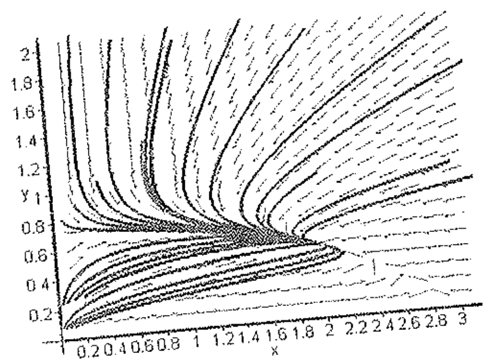
$$\mathbf{J}(2,1/2) = \begin{pmatrix} -1 & -2 \\ -1/16 & -1/2 \end{pmatrix}.$$

The eigenvalues and eigenvectors are

$$r_1 = \frac{-3+\sqrt{3}}{4}\,,\ \xi^{(1)} = \begin{pmatrix} 1 \\ -\frac{1+\sqrt{3}}{8} \end{pmatrix}\ ;\ r_2 = -\frac{3+\sqrt{3}}{4}\,,\ \xi^{(2)} = \begin{pmatrix} 0 \\ \frac{-1+\sqrt{3}}{8} \end{pmatrix}.$$

The eigenvalues are negative, hence the critical point $(2,1/2)$ is a stable node, which is asymptotically stable.

(d,e)



(f) Except for initial conditions along the coordinate axes, almost all solutions converge to the stable node $(2,1/2)$.

7. It follows immediately that

$$\begin{aligned}(\sigma_1 X + \sigma_2 Y)^2 - 4\sigma_1\sigma_2 XY &= \sigma_1^2 X^2 + 2\sigma_1\sigma_2 XY + \sigma_2^2 Y^2 - 4\sigma_1\sigma_2 XY \\ &= (\sigma_1 X - \sigma_2 Y)^2.\end{aligned}$$

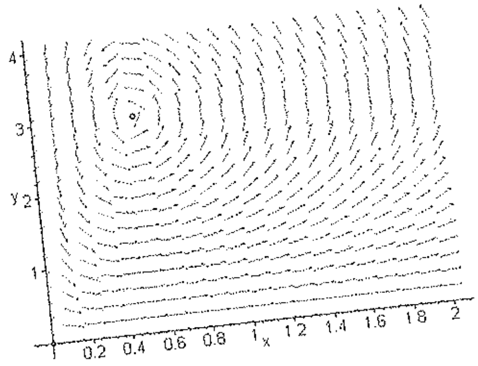
Since all parameters and variables are positive, it follows that

$$(\sigma_1 X + \sigma_2 Y)^2 - 4(\sigma_1\sigma_2 - \alpha_1\alpha_2)XY \geq 0\,.$$

Hence the radicand in Eq.(39) is nonnegative.

9.5

1.(a)



(b) The critical points are solutions of the system of equations

$$x(1.5 - 0.5\,y) = 0$$
$$y(-0.5 + x) = 0\,.$$

The two critical points are $(0\,,0)$ and $(0.5\,,3)$.

(c) The Jacobian matrix of the vector field is

$$\mathbf{J} = \begin{pmatrix} 3/2 - y/2 & -x/2 \\ y & -1/2 + x \end{pmatrix}.$$

At the critical point $(0\,,0)$, the coefficient matrix of the linearized system is

$$\mathbf{J}(0\,,0) = \begin{pmatrix} 3/2 & 0 \\ 0 & -1/2 \end{pmatrix}.$$

The eigenvalues and eigenvectors are

$$r_1 = 3/2\;,\; \xi^{(1)} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}\;;\; r_2 = -1/2\;,\; \xi^{(2)} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

The eigenvalues are of opposite sign, hence the origin is a saddle, which is unstable. At the critical point $(0.5\,,3)$, the coefficient matrix of the linearized system is

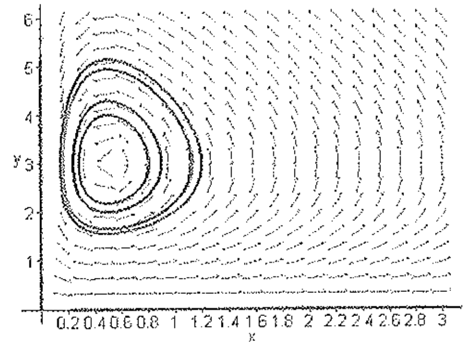
$$\mathbf{J}(0.5\,,3) = \begin{pmatrix} 0 & -1/4 \\ 3 & 0 \end{pmatrix}.$$

The eigenvalues and eigenvectors are

$$r_1 = i\,\frac{\sqrt{3}}{2}\;,\; \xi^{(1)} = \begin{pmatrix} 1 \\ -2\,i\sqrt{3} \end{pmatrix}\;;\; r_2 = -i\,\frac{\sqrt{3}}{2}\;,\; \xi^{(2)} = \begin{pmatrix} 1 \\ 2\,i\sqrt{3} \end{pmatrix}.$$

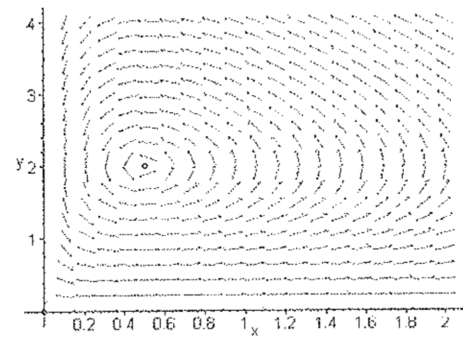
The eigenvalues are purely imaginary. Hence the critical point is a center, which is stable.

(d,e)



(f) Except for solutions along the coordinate axes, almost all trajectories are closed curves about the critical point $(0.5\,,3)$.

2.(a)



(b) The critical points are the solution set of the system of equations

$$x(1-0.5\,y)=0$$
$$y(-0.25+0.5\,x)=0\,.$$

The two critical points are $(0\,,0)$ and $(0.5\,,2)$.

(c) The Jacobian matrix of the vector field is

$$\mathbf{J}=\begin{pmatrix}1-y/2 & -x/2\\ y/2 & -1/4+x/2\end{pmatrix}.$$

At the critical point $(0\,,0)$, the coefficient matrix of the linearized system is

$$\mathbf{J}(0\,,0)=\begin{pmatrix}1 & 0\\ 0 & -1/4\end{pmatrix}.$$

The eigenvalues and eigenvectors are

$$r_1=1\ ,\ \boldsymbol{\xi}^{(1)}=\begin{pmatrix}1\\0\end{pmatrix}\ ;\ r_2=-1/4\ ,\ \boldsymbol{\xi}^{(2)}=\begin{pmatrix}0\\1\end{pmatrix}.$$

The eigenvalues are of opposite sign, hence the origin is a saddle, which is unstable. At the critical point $(0.5\,,2)$, the coefficient matrix of the linearized system is

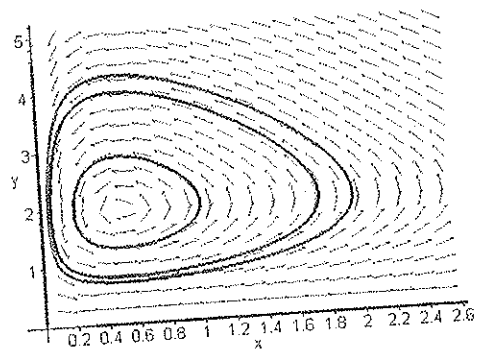
$$\mathbf{J}(0.5\,,2) = \begin{pmatrix} 0 & -1/4 \\ 1 & 0 \end{pmatrix}.$$

The eigenvalues and eigenvectors are

$$r_1 = i/2\;,\; \boldsymbol{\xi}^{(1)} = \begin{pmatrix} 1 \\ -2\,i \end{pmatrix}\;;\; r_2 = -i/2\;,\; \boldsymbol{\xi}^{(2)} = \begin{pmatrix} 1 \\ 2\,i \end{pmatrix}.$$

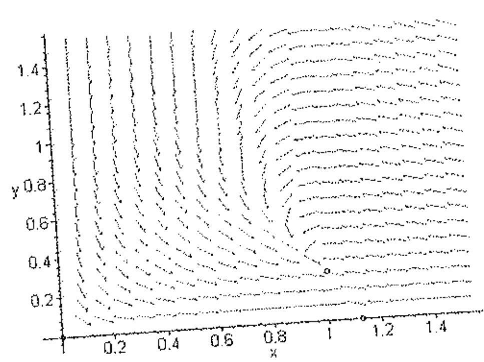
The eigenvalues are purely imaginary. Hence the critical point is a center, which is stable.

(d,e)



(f) Except for solutions along the coordinate axes, almost all trajectories are closed curves about the critical point $(0.5\,,2)$.

4.(a)



(b) The critical points are the solution set of the system of equations

$$x(9/8 - x - y/2) = 0$$
$$y(-1 + x) = 0\,.$$

The three critical points are $(0\,,0)$, $(9/8\,,0)$ and $(1\,,1/4)$.

(c) The Jacobian matrix of the vector field is

$$\mathbf{J} = \begin{pmatrix} 9/8 - 2x - y/2 & -x/2 \\ y & -1 + x \end{pmatrix}.$$

At the critical point $(0\,,0)$, the coefficient matrix of the linearized system is

$$\mathbf{J}(0\,,0) = \begin{pmatrix} 9/8 & 0 \\ 0 & -1 \end{pmatrix}.$$

The eigenvalues and eigenvectors are

$$r_1 = 9/8\;,\; \xi^{(1)} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}\;;\; r_2 = -1\;,\; \xi^{(2)} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

The eigenvalues are of opposite sign, hence the origin is a saddle, which is unstable. At the critical point $(9/8\,,0)$, the coefficient matrix of the linearized system is

$$\mathbf{J}(9/8\,,0) = \begin{pmatrix} -9/8 & -9/16 \\ 0 & 1/8 \end{pmatrix}.$$

The eigenvalues and eigenvectors are

$$r_1 = -\frac{9}{8}\;,\; \xi^{(1)} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}\;;\; r_2 = \frac{1}{8}\;,\; \xi^{(2)} = \begin{pmatrix} 9 \\ -20 \end{pmatrix}.$$

The eigenvalues are of opposite sign, hence the critical point $(9/8\,,0)$ is a saddle, which is unstable.
At the critical point $(1\,,1/4)$, the coefficient matrix of the linearized system is

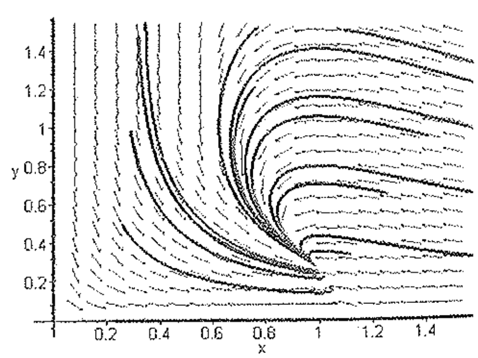
$$\mathbf{J}(1\,,1/4) = \begin{pmatrix} -1 & -1/2 \\ 1/4 & 0 \end{pmatrix}.$$

The eigenvalues and eigenvectors are

$$r_1 = \frac{-2+\sqrt{2}}{4}\;,\; \xi^{(1)} = \begin{pmatrix} -2+\sqrt{2} \\ 1 \end{pmatrix}\;;\; r_2 = \frac{-2-\sqrt{2}}{4}\;,\; \xi^{(2)} = \begin{pmatrix} -2-\sqrt{2} \\ 1 \end{pmatrix}.$$

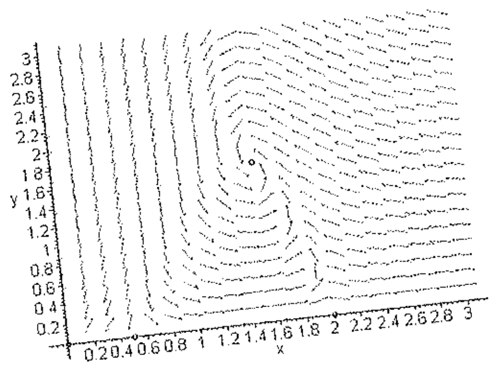
The eigenvalues are both negative. Hence the critical point is a stable node, which is asymptotically stable.

(d,e)



(f) Except for solutions along the coordinate axes, all solutions converge to the critical point $(1\,,1/4)$.

5.(a)



(b) The critical points are solutions of the system of equations

$$x(-1+2.5\,x-0.3\,y-x^2)=0$$
$$y(-1.5+x)=0\,.$$

The four critical points are $(0\,,0)$, $(1/2\,,0)$, $(2\,,0)$ and $(3/2\,,5/3)$.

(c) The Jacobian matrix of the vector field is

$$\mathbf{J}=\begin{pmatrix}-1+5x-3x^2-3y/10 & -3x/10\\ y & -3/2+x\end{pmatrix}.$$

At the critical point $(0\,,0)$, the coefficient matrix of the linearized system is

$$\mathbf{J}(0\,,0)=\begin{pmatrix}-1 & 0\\ 0 & -3/2\end{pmatrix}.$$

The eigenvalues and eigenvectors are

$$r_1=-1\;,\;\xi^{(1)}=\begin{pmatrix}1\\0\end{pmatrix}\;;\;r_2=-3/2\;,\;\xi^{(2)}=\begin{pmatrix}0\\1\end{pmatrix}.$$

The eigenvalues are both negative, hence the critical point $(0\,,0)$ is a stable node, which is asymptotically stable.
At the critical point $(1/2\,,0)$, the coefficient matrix of the linearized system is

$$\mathbf{J}(1/2\,,0)=\begin{pmatrix}3/4 & -3/20\\ 0 & -1\end{pmatrix}.$$

The eigenvalues and eigenvectors are

$$r_1=\frac{3}{4}\;,\;\xi^{(1)}=\begin{pmatrix}1\\0\end{pmatrix}\;;\;r_2=-1\;,\;\xi^{(2)}=\begin{pmatrix}3\\35\end{pmatrix}.$$

The eigenvalues are of opposite sign, hence the critical point $(1/2\,,0)$ is a saddle, which is unstable.