PREDATOR-PREY PROBLEMS

x = predator, y = prey
$\alpha, \beta, a, b, c, d > 0$

THERE ARE 3 TYPES OF MODELS

$x \geq 0$, $y \geq 0$

(1) P-PREY WITH OVERCROWDING

$$x' = (-\alpha - ax + by)x$$
$$y' = (\beta - cx - dy)y$$

$a = 0, d = 0 \rightarrow$ infinite carrying capacity.

(2) COMPETITION between two species

$$x' = (\alpha - ax - by)x$$
$$y' = (\beta - cx - dy)y$$

(3) CO-OPERATION between two species

$$x' = (\alpha - ax + by)x$$
$$y' = (\beta + cx - dy)y$$

<u>OUTLINE</u> EACH PROBLEM HAS THE FORM

$$x' = f(x,y)$$
$$y' = g(x,y)$$

(i) FIND EQUILIBRIA: THEY SATISFY $f(x,y) = 0$, $g(x,y) = 0$

(ii) let (x_0, y_0) BE AN EQUILIBRIA. LET

$$x = x_0 + \hat{x}, \quad y = y_0 + \hat{y}.$$

LINEARIZING we get

$$\begin{pmatrix} \hat{x}' \\ \tilde{y}' \end{pmatrix} = \begin{pmatrix} f_x^0 & f_y^0 \\ g_x^0 & g_y^0 \end{pmatrix} \begin{pmatrix} \tilde{x} \\ \tilde{y} \end{pmatrix} \qquad J = \begin{pmatrix} f_x^0 & f_y^0 \\ g_x^0 & g_y^0 \end{pmatrix}$$

(iii) FOR each eq. point calculate eigenvalues of J AND classify stability (saddle, spiral, center ...). DRAW local trajectories.

(iv) plot nullclines to get vector field. plot $f(x,y) = 0$, $g(x,y) = 0$ AND find regions where $x' > 0$, $x' < 0$, $y' > 0$, $y' < 0$.

(v) DRAW the entire phase-plane and interpret biologically.

EXAMPLE 1 (LOTTKA-VOLTERRA)

$$x' = (-4+y)x = f(x,y) \qquad x\text{: predator, } y\text{: prey}$$

$$y' = (4-x)y = g(x,y) \qquad \text{infinite carrying capacity}$$

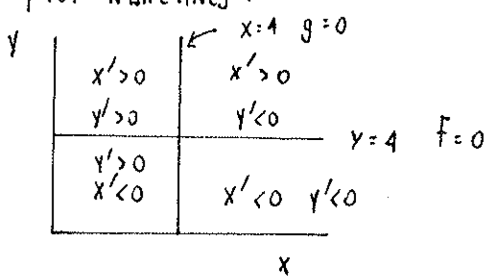
EQUILIBRIA AT $\begin{array}{l} x(y-4)=0 \\ y(4-x)=0 \end{array} \longrightarrow (4,4),\ (0,0)$

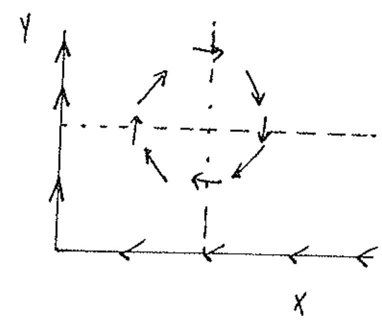
NOW $J = \begin{pmatrix} f_x^0 & f_y^0 \\ g_x^0 & g_y^0 \end{pmatrix} = \begin{pmatrix} -4+y_0 & x_0 \\ -y_0 & 4-x_0 \end{pmatrix}$

AT $(0,0)$ $\quad J = \begin{pmatrix} -4 & 0 \\ 0 & 4 \end{pmatrix}$ eigenvalues are $\lambda_1 = 4,\ \lambda_2 = -4 \longrightarrow$ saddle point

AT $(4,4)$ $\quad J = \begin{pmatrix} 0 & 4 \\ -4 & 0 \end{pmatrix}$ eigenvalues are $\lambda = \pm 4i$ center.

NOW we plot nullclines:





DO WE HAVE closed orbits? centers are not robust to small perturbations.
FIND a conserved integral.

$$\frac{dy}{dx} = \frac{(4-x)y}{(y-4)x} \qquad \text{THUS} \quad \left(\frac{y-4}{y}\right)dy = \left(\frac{4-x}{x}\right)dx \longrightarrow y - 4\ln|y| = 4\ln|x| - x + E$$

SO THAT $\quad E = y + x - 4\ln(xy) \qquad$ E is a constant independent of t.

WE CAN WRITE THIS AS $\quad E = \ln\left[\dfrac{e^{x+y}}{x^4y^4}\right]$

HENCE $\quad (x^4e^{-x})(y^4e^{-y}) = C \qquad$ FOR EACH $x>0 \longrightarrow \exists$ two values of y

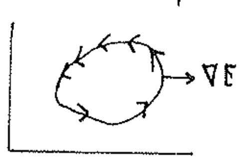
FOR EACH $y>0 \longrightarrow \exists$ two values of x.

NOTICE $\displaystyle\max_{x\geq 0} x^4e^{-x} = 4^4e^{-4}$ which occurs when $x=4$.

HENCE $\quad x^4y^4e^{-(x+y)} = C \quad$ WITH $\quad C \leq 2\cdot 4^4(e^{-4})^2 \longrightarrow$ PERIOD SOLUTIO.

AND we have closed orbits in the phase plane.

NOW NOTICE $\nabla E \cdot \underline{x}' = 0$



$\nabla E = \left(1 - 4/x,\ 1 - 4/y\right)$

$\nabla E \cdot \underline{x}' = \underline{0}$ can be checked.

REMARK
$$\frac{dx}{dt} = (-a + by)x \qquad x = \text{predator}$$
$$\frac{dy}{dt} = (c - dx)y \qquad y = \text{prey}$$

$$J = \begin{pmatrix} -a + by_0 & bx_0 \\ -dy_0 & c - dx_0 \end{pmatrix}$$

equilibria is $x_0 = c/d$ AND $y_0 = a/b$. $\rightarrow$ $J = \begin{pmatrix} 0 & bc/d \\ -ad/b & 0 \end{pmatrix}$

eigenvalues are $\lambda = \pm i(ac)^{1/2}$ frequency near (x_0, y_0) is $(ac)^{1/2}$.

EXAMPLE 2 (COOPERATION ~~COMPETITION~~ model)

$$x' = (4 - 2x + y)x = f(x,y) \qquad (4-2x+y)x = 0$$
$$y' = (4 + x - 2y)y = g(x,y) \qquad \text{AND} \quad (4+x-2y)y = 0$$
$\Big\}$ eq. equations

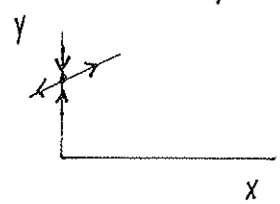
EQ. POINTS $(0,0)$, $(0,2)$, $(2,0)$, $(4,4)$

$$J = \begin{pmatrix} 4 - 4x_0 + y_0 & x_0 \\ y_0 & 4 + x_0 - 4y_0 \end{pmatrix}$$

NEAR $(0,2)$ $\quad J = \begin{pmatrix} 6 & 0 \\ 2 & -4 \end{pmatrix}$ eigenvalues $\lambda_1 = -4$, $\lambda_2 = 6$ saddle point

$\lambda_1 = -4 \quad (J - \lambda_1 I)\underline{v}_1 = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \rightarrow \begin{pmatrix} 10 & 0 \\ 20 & 0 \end{pmatrix} \underline{v}_1 = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \rightarrow \underline{v}_1 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

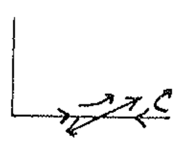
$\lambda_2 = 6 \quad (J - \lambda_2 I)\underline{v}_2 = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \rightarrow \begin{pmatrix} 0 & 0 \\ 20 & -10 \end{pmatrix} \underline{v}_2 = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \rightarrow \underline{v}_2 = \begin{pmatrix} 5 \\ 1 \end{pmatrix}$



NEAR $(2,0)$ $\quad J = \begin{pmatrix} -4 & 2 \\ 0 & 6 \end{pmatrix}$ eigenvalues $\lambda_1 = -4$, $\lambda_2 = 6$ saddle point

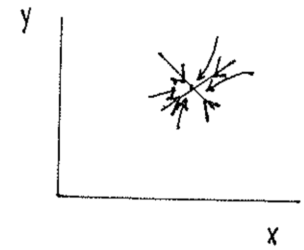
$\lambda_1 = -4 \rightarrow (J - \lambda_1 I)\underline{v}_1 = \underline{0} \rightarrow \begin{pmatrix} 0 & 2 \\ 0 & 10 \end{pmatrix} \underline{v}_1 = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad \underline{v}_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$

$\lambda_2 = 6 \rightarrow (J - \lambda_2 I)\underline{v}_2 = \underline{0} \rightarrow \begin{pmatrix} -10 & 2 \\ 0 & 0 \end{pmatrix} \underline{v}_2 = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad v_2 = \begin{pmatrix} 1 \\ 5 \end{pmatrix}$



NEAR (4,4) $\quad J = \begin{pmatrix} -8 & 4 \\ 4 & -8 \end{pmatrix}$ $\qquad (-8-\lambda)^2 - 16 = 0$

$-8-\lambda = \pm 4 \qquad \lambda_2 = -12,\ \lambda_1 = -4$ stable node



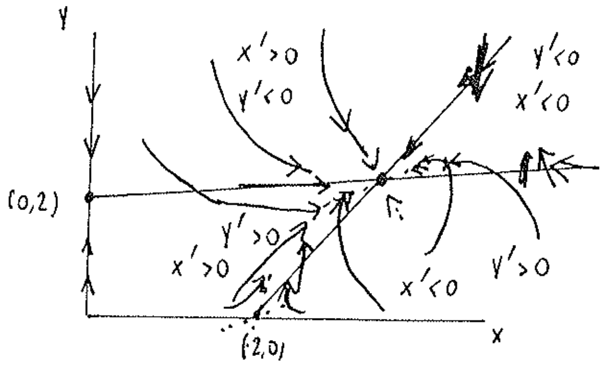
$\lambda_1 = -4: \ (J - \lambda_1 I)\underline{v}_1 = \underline{0} \longrightarrow \begin{pmatrix} -4 & 4 \\ 4 & -4 \end{pmatrix} \underline{v}_1 = \underline{0} \longrightarrow \underline{v}_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

$\lambda_2 = -12: \ (J - \lambda_2 I)\underline{v}_1 = \underline{0} \longrightarrow \begin{pmatrix} 4 & 4 \\ 4 & 4 \end{pmatrix} \underline{v}_2 = \underline{0} \longrightarrow \underline{v}_2 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$

THUS $\quad \underline{x}(t) = c_1 e^{-4t}\underline{v}_1 + c_2 e^{-12t}\underline{v}_2$ NEAR (4,4)

$\underline{x}(t) \longrightarrow c_1 e^{-4t}\underline{v}_1$ as $t \to \infty$ unless $c_1 = 0$ (smallest eigenvalue gives deca

NULLCLINES

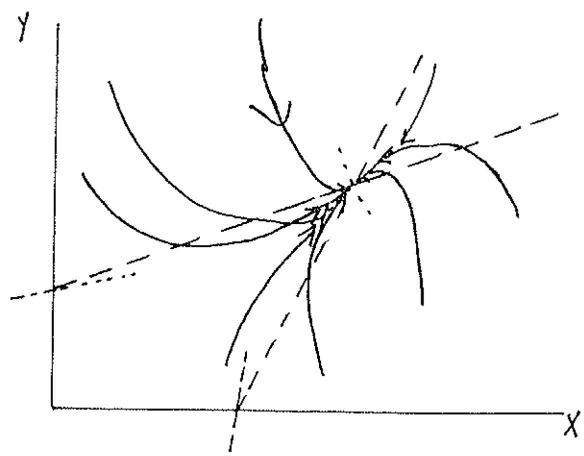


$f = 0$ ON $x = 0, \quad y = 2x - 4$

$g = 0$ ON $y = 0, \quad y = 2 + x/2$

THE FLOW IS computed ON Next page.

Biologically as $t \to \infty$ then $(x, y) \longrightarrow (4,4)$ As $t \to \infty$ both species can co-exist.



stable and unstable manifolds do not follow the nullcline but are between nullclines.

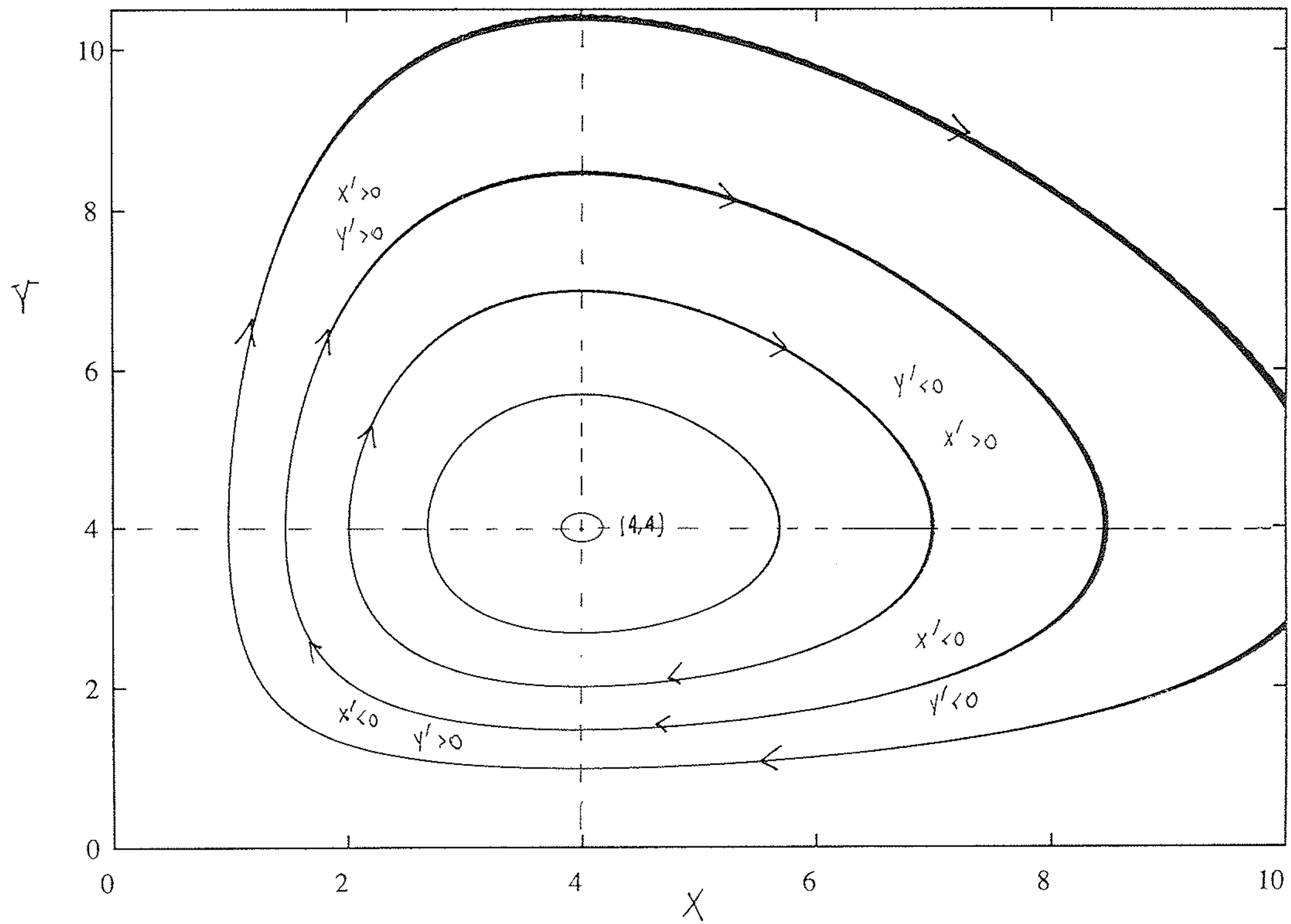
LOTTKA- VOLTERRA SYSTEM
(NO OVERCROWDING EFFECT)

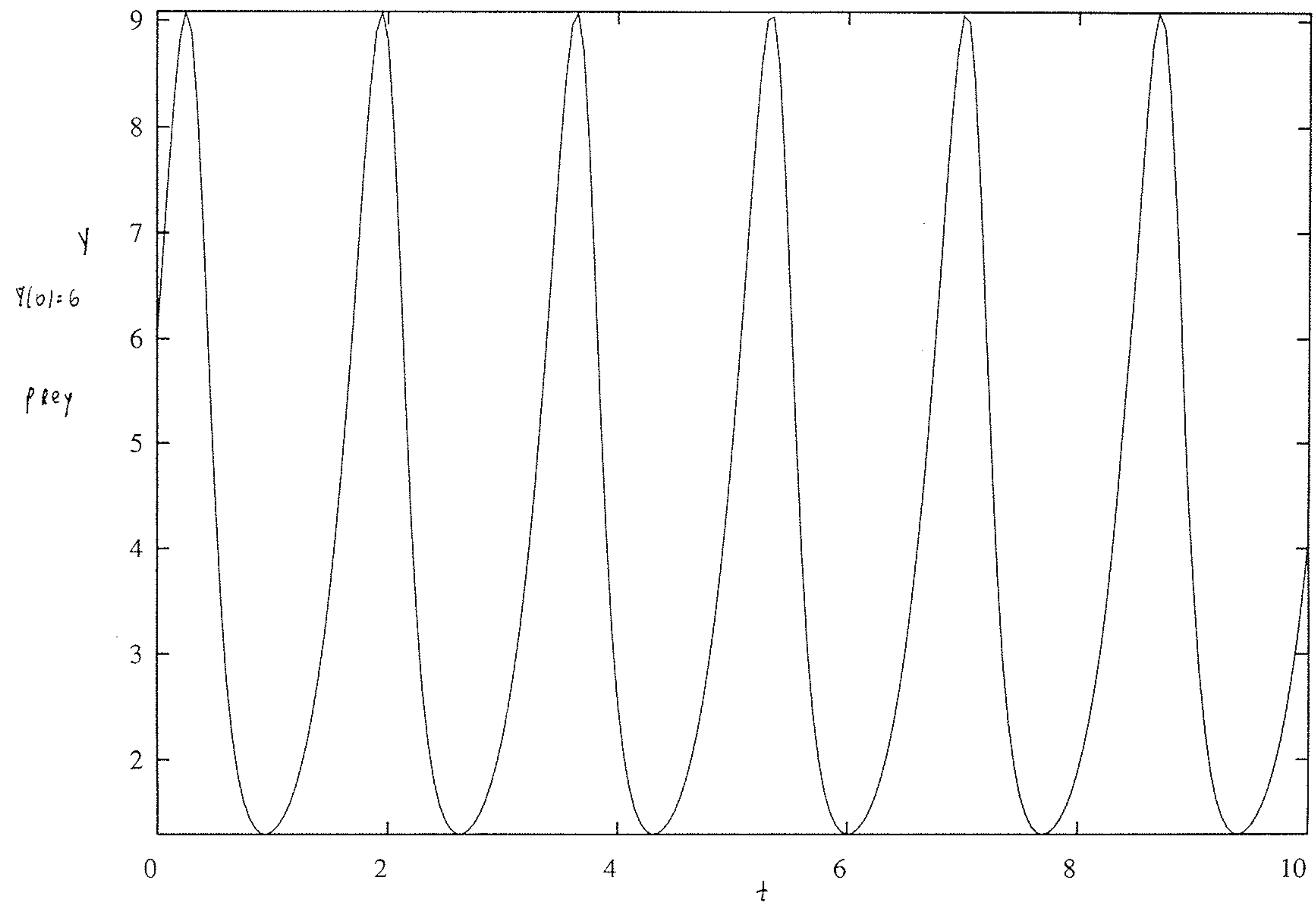
$x' = (-4 + y)x$

$y' = (4 - x)y$

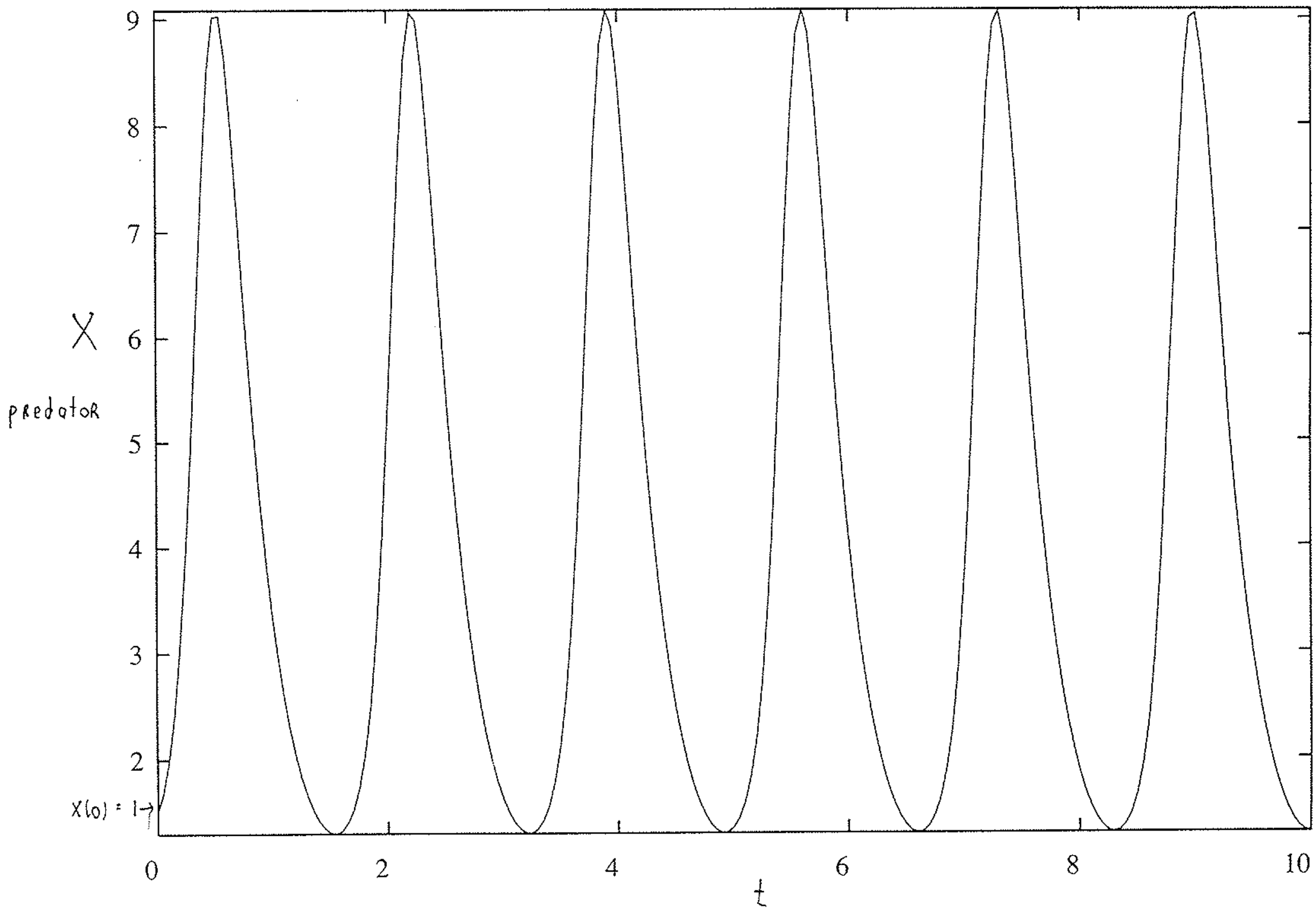
PERIODIC SOLUTIONS

X: PREDATOR Y: PREY.





y
y(0)=6
prey
9
8
7
6
5
4
3
2
0
2
4
6
8
10
t



X
predator
X(0) = 1→
9
8
7
6
5
4
3
2
0
2
4
6
8
10
t

EXAMPLE 3 (STABLE COMPETITION)

$$x' = (2-2x-y)\,x = f(x,y)$$
$$y' = (2-x-2y)\,y = g(x,y)$$

EQUILIBRIUM POINTS ARE AT $(1,0), (0,1), (0,0), (2/3, 2/3)$

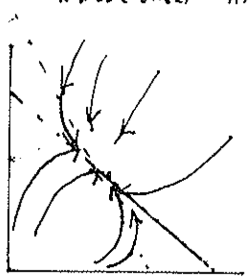
$$J = \begin{pmatrix} 2-4x_0-y_0 & -x_0 \\ -y_0 & 2-x_0-4y_0 \end{pmatrix}$$

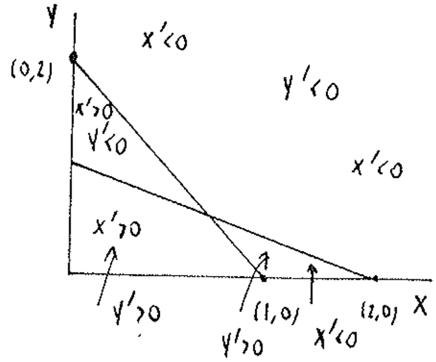
AT $(x_0, y_0) = (2/3, 2/3)$ $\quad J = \begin{pmatrix} -4/3 & -2/3 \\ -2/3 & -4/3 \end{pmatrix}$ $\quad \mathrm{tr}\, J = -8/3 < 0 \quad \mathrm{RE}[\lambda] < 0$

$$\det J = \frac{16}{9} - \frac{4}{9} = \frac{12}{9} > 0$$

THUS SINCE $\mathrm{tr}\, J = \lambda_1 + \lambda_2$ $\det J = \lambda_1 \lambda_2$ $\longrightarrow$ either eigenvalues are both real negative or have negative real parts. BUT J is symmetric $\longrightarrow$ both eigenvalues real negative. stable node

NULLCLINES AS SHOWN





$f=0 \quad x=0,\ y = 2-2x$

$g=0 \quad y=0 \quad y = 1 - x/2$

$\lim_{t\to\infty} (x,y) = (2/3, 2/3)$ for almost all initial conditions.

EXAMPLE (UNSTABLE COMPETITION)

$$x' = (2-x-2y)\,x = f(x,y)$$
$$y' = (2-y-2x)\,y = g(x,y)$$

HERE BOTH species will NOT co-exist as $t \to \infty$.
either $\lim_{t\to\infty} (x,y) = (2,0)$ OR $(0,2)$ depending on initial condition

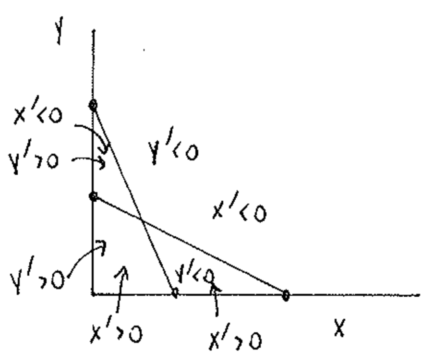
EQUILIBRIA AT $(2,0), (0,2), (0,0), (2/3, 2/3)$.

$$J = \begin{pmatrix} 2-2x_0-2y_0 & -2x_0 \\ -2y_0 & 2-2y_0-2x_0 \end{pmatrix} \qquad \text{AT } (2/3, 2/3) \quad J = \begin{pmatrix} -2/3 & -4/3 \\ -4/3 & -2/3 \end{pmatrix}$$

trace $J = -4/3$ $\quad \det J = -12/9 < 0$.

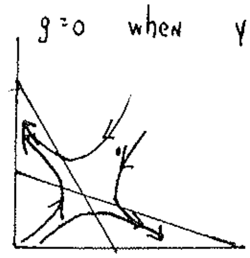
THUS $\lambda_1 > 0$, $\lambda_2 < 0$ Real. we have a saddle point (unstable).

NULLCLINES

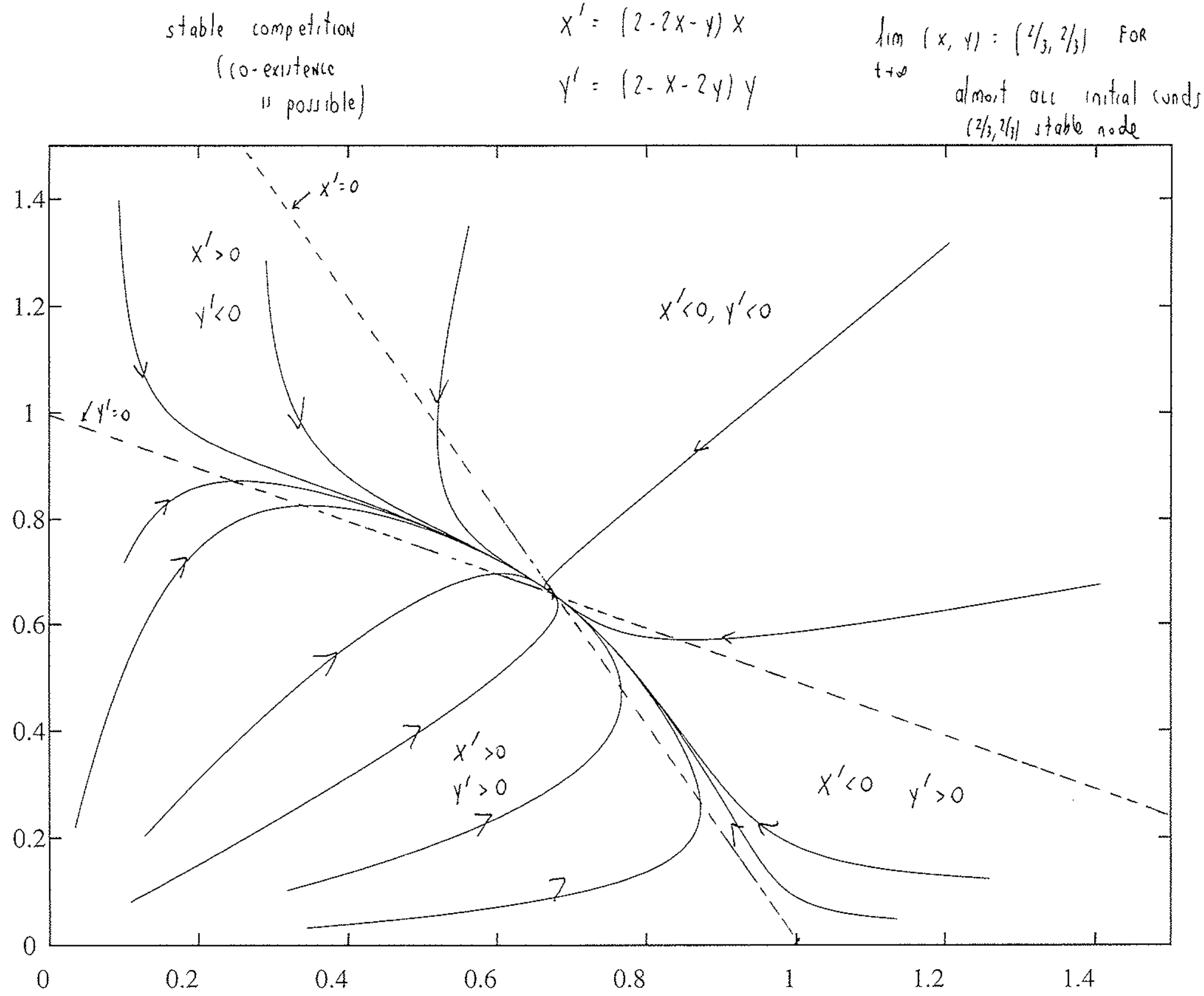


$f=0$ when $x=0,\ y = 1 - x/2$

$g=0$ when $y=0,\ y = 2-2x$



picture is on next page


stable competition
(co-existence is possible)
$x' = (2-2x-y)x$
$y' = (2-x-2y)y$
$\lim_{t\to\infty} (x, y) = (2/3, 2/3)$ for almost all initial conds.
$(2/3, 2/3)$ stable node
$x'=0$
$y'=0$
$x'>0$ $y'<0$
$x'<0$, $y'<0$
$x'>0$ $y'>0$
$x'<0$ $y'>0$
0
0.2
0.4
0.6
0.8
1
1.2
1.4

competitive exclusion:

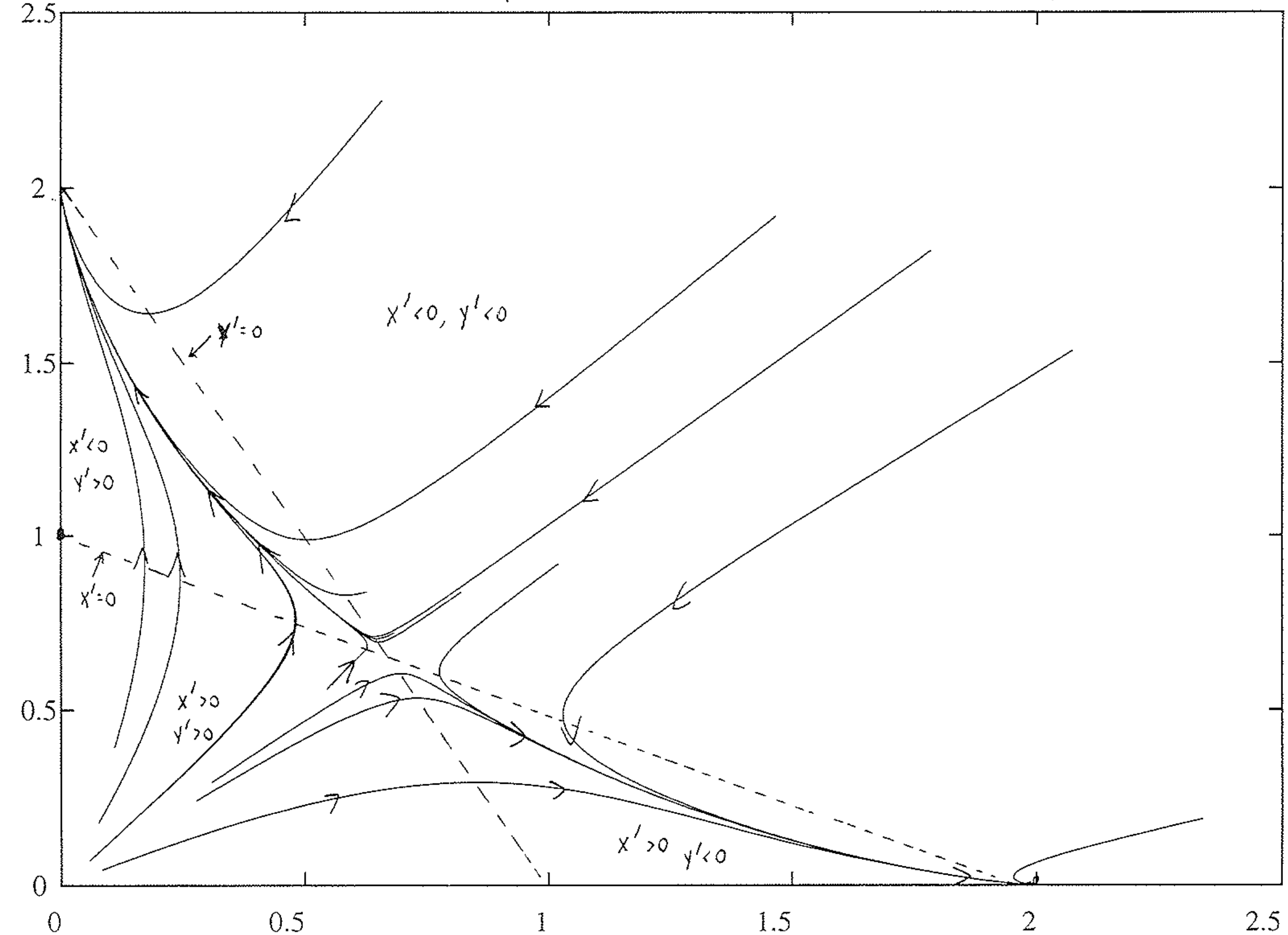
stable - coexistence unlikely.

$x' = (2 - x - 2y)x$

$y' = (2 - y - 2x)y$

$\lim_{t\to\infty} (x, y) = (2, 0)$ or $(0, 2)$ depending on initial cond.

$(2/3, 2/3)$ is a saddle point.



COOPERATION MODEL

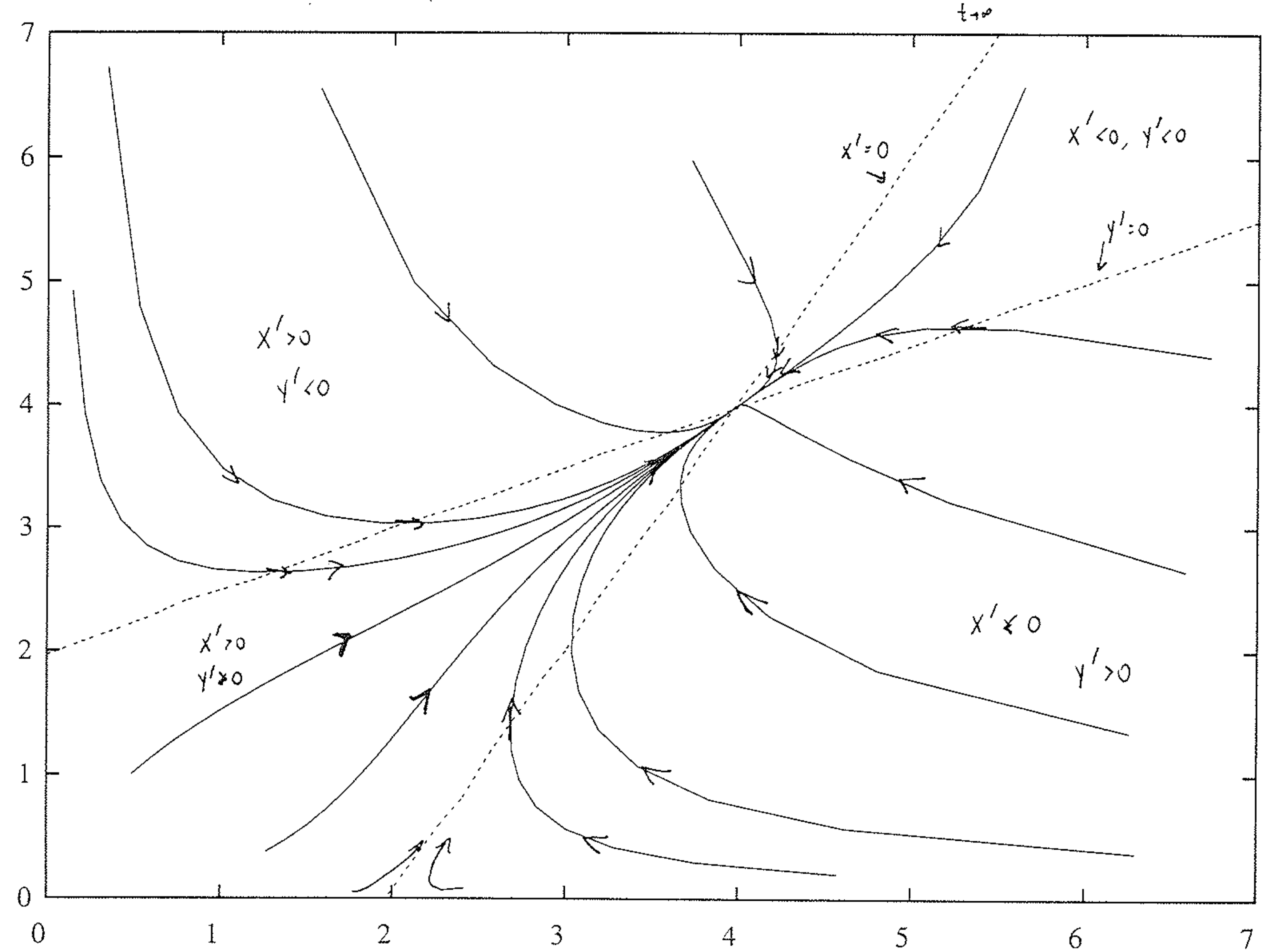
$x' = (4 - 2x + y)x$

$y' = (4 + x - 2y)y$

EQ. POINTS $(0,0), (4,4), (0,2), (2,0)$

$(4,4)$ is a stable node.

$\lim_{t\to\infty} (x(t), y(t)) = (4,4)$.



EXAMPLE (PREDICTION OF BEHAVIOR)

$$N_1' = r N_1 \left(1 - N_1/K_1\right) - b_1 N_1 N_2 \qquad N_1' = dN_1/dt$$

$$N_2' = r_2 N_2 \left(1 - N_2/K_2\right) - b_2 N_1 N_2$$

For $r_i > 0$, $K_i > 0$, $b_i > 0$ this is a competition model. We non-dimensionalize by setting $N_1 = K_1 x$ $\quad t = T\tau$ $\quad N_2 = \gamma y$

$$\frac{K_1}{T} x' = r_1 K_1 x (1 - x) - b_1 K_1 \gamma x y \longrightarrow \frac{x'}{r_1 T} = x(1-x) - \frac{b_1 \gamma}{r_1} x y$$

$$\frac{\gamma}{T} y' = r_2 \gamma y \left(1 - y/(K_2/\gamma)\right) - b_2 K_1 \gamma x y \longrightarrow y' = r_2 T y \left(1 - y/(K_2/\gamma)\right) - b_2 K_1 T x y.$$

choose $T = 1/r_1$ AND $\gamma = (b_1/r_1)^{-1} = r_1/b_1$

let's get

$$x' = x(1-x) - xy \qquad a = r_2/r_1$$

$$y' = a y (1 - b y) - c x y \qquad b = \frac{\gamma}{K_2} = \frac{b_1^{-1}}{r_1^{-1} K_2} = \frac{r_1}{K_2 b_1}$$

$$c = b_2 K_1 / r_1$$

EQUILIBRIA ARE:

$$y = 1 - x$$

$$a(1 - b y) = c x \longrightarrow y = \frac{1}{b} - \frac{c}{ab} x$$

$$x' = x\left[1 - x - y\right]$$

$$y' = y\left[a - aby - cx\right]$$

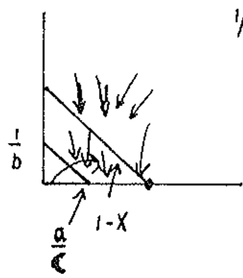
equilibria at $y = 1/b, x = 0$

$y = 0, x = 1$

AT $(0,0)$ AND maybe one more

Four relevant pictures:

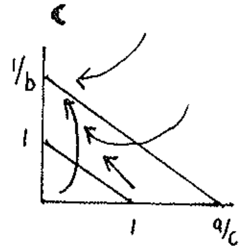
① x wins



$1/b < 1$, $\frac{a}{c} < 1$

$\lim_{t \to \infty} (x,y) = (1,0)$

② y wins



$1/b > 1$, $a/c > 1$

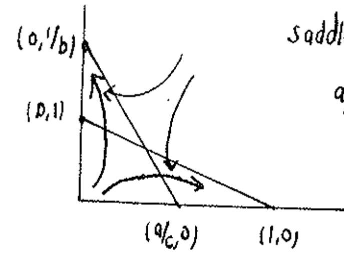
$\lim_{t \to \infty} (x,y) = (0, 1/b)$

co-existence if $b > 1$; $a/c > 1$

$\longrightarrow r_1/K_2 b_1 > 1, \quad r_2/b_2 K_2 > 1$

growth rate exceeds carrying capacity for co-existence

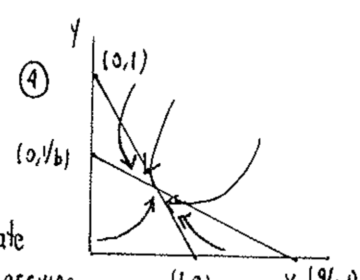
③ either x or y wins



saddle point

$a/c < 1$, $\frac{1}{b} >$

④



co-existence $1/b < 1$

$a/c > 1$.

CONSERVATIVE SYSTEMS

CONSIDER SYSTEMS OF THE FORM

$$x'' = F(x)$$

MULTIPLY BY x' TO GET

$$x' x'' = x' F(x)$$

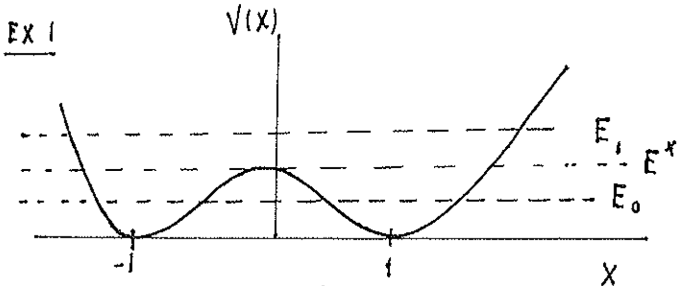
LET $V(x) = -\int^x F(\lambda)\, d\lambda$. THEN, $dV/dt = -F'(x)\, dx/dt$. $\quad V'(x) = -F(x)$

THUS,

$$\frac{d}{dt}\left[\tfrac{1}{2}x'^2 + V(x)\right] = 0$$

HENCE $\quad \frac{1}{2}x'^2 + V(x) = E \qquad E$ total energy.

EX 1



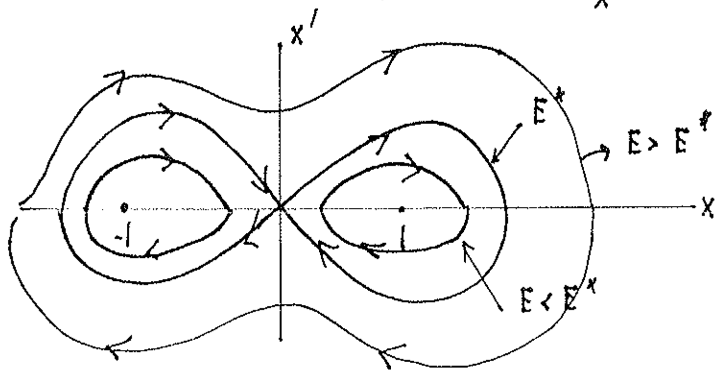
$$V(x) = \frac{1}{4}(1-x^2)^2$$

$$V'(x) = -F(x)$$

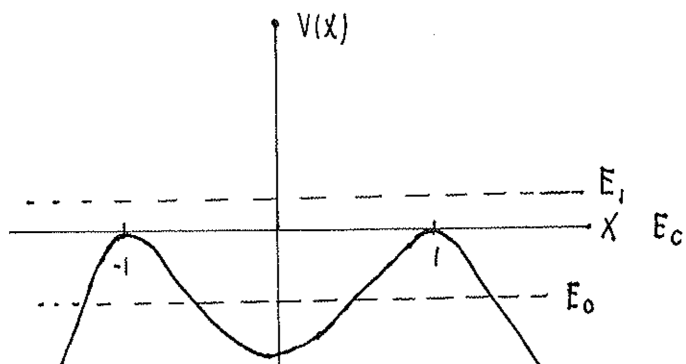
$$\longrightarrow F(x) = x(1-x^2)$$

$$x'' = x(1-x^2)$$

$$x'' - x + x^3 = 0$$

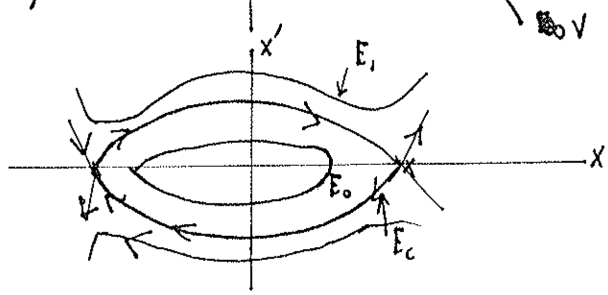


EX 2 $\quad$ LET $F(x) = -x + x^3 \longrightarrow V(x) = -\frac{1}{4}(1-x^2)^2$



closed orbits are periodic solutions.

heteroclinic connection $-1 \longrightarrow 1$.



NOTE:

(i) symmetry $\dot{x} \longrightarrow -\dot{x}$

EQUILIBRIA SUPPOSE THAT $V'(X_E) = 0$.

WE LET $X = X_E + \tilde{X}$ WITH $\tilde{X} \ll 1$ TO GET

$$\tilde{X}'' = -V'[X_E] - \hat{X} V''(X_E) + \dots$$

THUS $\hat{X}'' + V''(X_E)\hat{X} = 0$

NOW $\hat{X} = e^{\lambda t} \longrightarrow \lambda^2 + V''(X_E) = 0$

(i) IF $V''(X_E) > 0 \rightarrow X_E$ IS LOCAL MINIMA OF V THEN

$\lambda = \pm i\left[V'(X_E)\right]^{1/2}$.

$\hat{X} \sim A\cos[w_E t] + B\sin[w_E t]$ $\quad w_E = \sqrt{V''(X_E)}$ frequency

local oscillations X_E is called a <u>center</u>

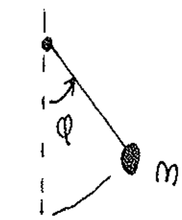
$\hat{X}$ is bounded as $t \to \infty \longrightarrow X_E$ is neutrally stable.

(ii) $V''(X_E) < 0 \longrightarrow \lambda = \pm\sqrt{-V''(X_E)}$

$$\hat{X} \sim A e^{\sqrt{-V''(X_E)}\,t} + B e^{-\sqrt{-V''(X_E)}\,t}$$

exponential growth and decay. X_E is called a saddle point.

EXAMPLE 1 (SIMPLE PENDULUM)



$$\varphi'' + \frac{g}{L}\sin\varphi = 0 \qquad V'(\varphi) = \frac{g}{L}\sin\varphi$$

$V'(\varphi) = \frac{g}{L}\sin\varphi \qquad V(\varphi) = \frac{g}{L}(1 - \cos\varphi)$

THUS $\dfrac{\varphi'^2}{2} + \dfrac{g}{L}(-\cos\varphi) = E$ $\qquad V(\varphi) = -\frac{g}{L}\cos\varphi$

NOW $V'(\varphi) = 0$ AT $\varphi = 0, \pm\pi, \pm 2\pi, \dots$

LET $\dfrac{\varphi'^2}{2} + \dfrac{g}{L}(-\cos\varphi) = E$

NOW $V''(\varphi) = \dfrac{g}{L}\cos\varphi = \dfrac{g}{L}\cos(n\pi) = \pm(-1)^n\dfrac{g}{L}$