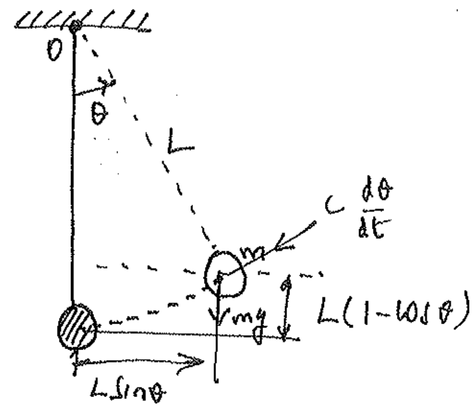
4/April/2012

(1)

Pendulum:-



Gravity : mg

Damping : $-c\left|\frac{d\theta}{dt}\right|$

Conservation of angular momentum:-

Rate of change of angular momentum about O

= Torque about O.

Total angular momentum about 'O' $= mL^2\left(\frac{d\theta}{dt}\right)$

$$\Rightarrow \quad mL^2\frac{d\theta}{dt} = -cL\left|\frac{d\theta}{dt}\right| - mgL\sin\theta$$

Assuming θ & $\frac{d\theta}{dt}$ to be positive, we get

$$\frac{d^2\theta}{dt^2} + \frac{c}{mL}\frac{d\theta}{dt} + \frac{g}{L}\sin\theta = 0$$

$$\Rightarrow \quad \boxed{\frac{d^2\theta}{dt^2} + \gamma\frac{d\theta}{dt} + \omega^2\sin\theta = 0}$$

: Nonlinear equation because of $\sin\theta$ term

~~For small~~

If $\theta << 1$; $\sin\theta \approx \theta$

In this case $\boxed{\frac{d^2\theta}{dt^2} + \gamma\frac{d\theta}{dt} + \omega^2\theta = 0}$: Linear 2nd order equation with constant coefficients.

Converting into two first order equations:

Let $x = \theta$

$y = \frac{d\theta}{dt}$

$$\Rightarrow \quad \frac{dx}{dt} = y$$

$$\frac{dy}{dt} = -\gamma y - \omega^2 \sin x$$

$$\Rightarrow \begin{bmatrix} \frac{dx}{dt} \\ \frac{dy}{dt} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & -\gamma \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} 0 \\ -\omega^2 \sin x \end{bmatrix}$$

Critical Points: $y = 0$

$\gamma y + \omega^2 \sin x = 0$

Using $y = 0$ in the second equation, $\omega^2 \sin x = 0$

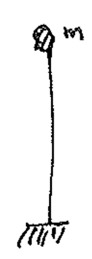
$\Rightarrow \sin x = 0$

$\Rightarrow x = 0, \pm\pi, \pm 2\pi, \ldots$

$\theta = 0$:

Stable

$\theta = \pi$:



Unstable

Consider the system

$$\frac{d}{dt}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & -\gamma \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} 0 \\ -\omega^2 \sin x \end{bmatrix}$$

$$\vec{X}' = A \vec{X} + \vec{g}(\vec{X})$$

$$A = \begin{bmatrix} 0 & 1 \\ 0 & -\gamma \end{bmatrix} \; ; \; \vec{g}(\vec{X}) = \begin{bmatrix} 0 \\ -\omega^2 \sin x \end{bmatrix} = \begin{bmatrix} g_1 \\ g_2 \end{bmatrix}$$

$$g_1 = 0$$
$$g_2 = -\omega^2 \sin x$$

Consider the origion $(0,0)$:-

$$\frac{g_2}{r} = \frac{-\omega^2 \sin x}{r}$$

As $x \to 0$;
$$\sin x = x - \frac{x^3}{3!} + \cdots$$

$$= \frac{-\omega^2}{r}\left(x - \frac{x^3}{3!} + \cdots\right)$$

If $x = r\cos\theta$; then

$$\frac{g_2}{r} = \frac{-\omega^2}{r}\left(r\cos\theta - \frac{r^3}{3!}\cos^3\theta + \cdots\right)$$

$$= -\omega^2\left(\cos\theta - \frac{r^2}{3!}\cos^3\theta + \cdots\right) \not\to 0$$

as $r \to 0$

What is wrong?

Rewriting the equations as follows:

$$\sin x = x + (\sin x - x)$$

$$\Rightarrow -\omega^2 \sin x = -\omega^2 x - \omega^2(\sin x - x)$$

$$\Rightarrow \frac{d}{dt}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -\omega^2 & -\gamma \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} 0 \\ -\omega^2(\sin x - x) \end{bmatrix}$$

$\therefore \quad \sin x - x = \frac{x^3}{3!} + \cdots$

$$\Rightarrow \quad \frac{g_2(x,y)}{r} = \frac{-r^3 \cos^3\theta}{3!} \omega^2 + \cdots$$

$\longrightarrow 0$ as as $r \to 0$.

$\Rightarrow$ Correct System:-

$$\frac{d}{dt}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -\omega^2 & -\gamma \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} 0 \\ -\omega^2(\sin x - x) \end{bmatrix}$$

$$\vec{X}' \qquad A \qquad \vec{X} \qquad \vec{g}$$

Near $(0,0)$:

$F(x,y) = y$

$G(x,y) = -\gamma y - \omega^2 \sin x$

$F_x = 0 \quad ; \quad F_y = 1$

$G_x = -\omega^2 \cos x \quad ; \quad G_y = -\gamma$

$$\frac{d}{dt}\begin{bmatrix} x-0 \\ y-0 \end{bmatrix} = \underbrace{\begin{bmatrix} 0 & 1 \\ -\omega^2 & -\gamma \end{bmatrix}}_{J}\begin{bmatrix} x-0 \\ y-0 \end{bmatrix}$$

$$J = \begin{bmatrix} 0 & 1 \\ -\omega^2 & -\gamma \end{bmatrix}$$

Eigenvalues: $|J - \lambda I| = 0 \Rightarrow \begin{vmatrix} 0-\lambda & 1 \\ -\omega^2 & -\gamma-\lambda \end{vmatrix} = 0$

$\Rightarrow -\lambda(-\gamma - \lambda) + \omega^2 = 0$

$\Rightarrow \lambda^2 + \gamma\lambda + \omega^2 = 0 \quad \Rightarrow \boxed{\lambda_{1,2} = \frac{-\gamma \pm \sqrt{\gamma^2 - 4\omega^2}}{2}}$

Case (I): If $r^2 - 4\omega^2 > 0$ $\Rightarrow$ Eigenvalues are real, unequal & negative

$\Rightarrow$ $(0,0)$ is asymptotically stable.

Case (II): If $r^2 - 4\omega^2 = 0$ $\Rightarrow$ Eigenvalues are real, equal & negative.

$\Rightarrow$ $(0,0)$ is asymptotically stable point of the linear system.

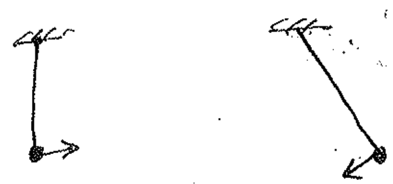
$\Rightarrow$ can be either a stable node or spiral point of the locally linear system.

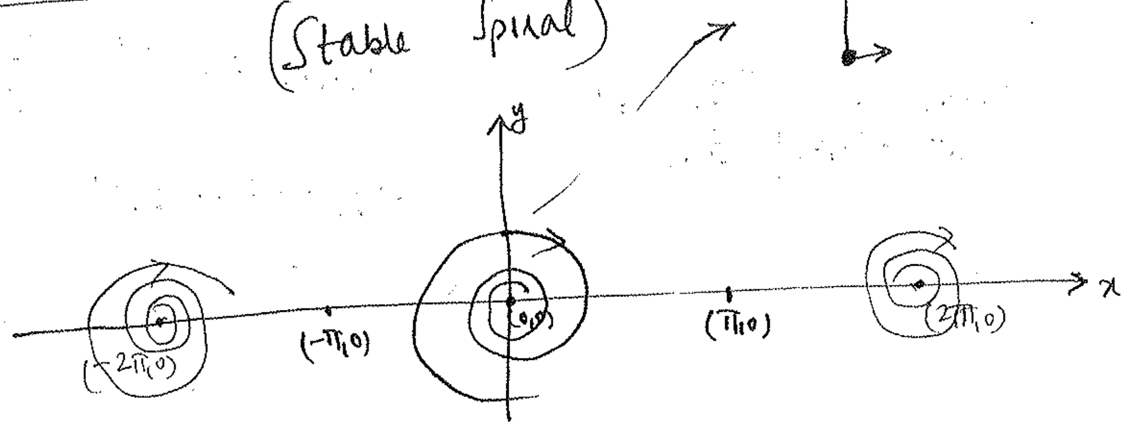
Case (III): $r^2 - 4w < 0$ $\Rightarrow$ Eigenvalues are complex with negative real part.

$\Rightarrow$ stable spiral of both linear and locally linear system.

$\therefore$ $(0,0)$ is either a stable node or a stable spiral.

Let us consider the case $r^2 - 4w < 0$ (Stable Spiral)





Near $(\pi, 0)$:

$$J = \begin{bmatrix} 0 & 1 \\ \omega^2 & -\gamma \end{bmatrix}$$

$$\frac{d}{dt}\begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ \omega^2 & -\gamma \end{bmatrix}\begin{bmatrix} u \\ v \end{bmatrix}$$ where $u = x - \pi$, $v = y$

Eigenvalues: $\begin{vmatrix} -\lambda & 1 \\ \omega^2 & -\gamma-\lambda \end{vmatrix} = 0$

$$\Rightarrow -\lambda(-\gamma-\lambda) - \omega^2 = 0$$

$$\Rightarrow \lambda^2 + \gamma\lambda - \omega^2 = 0$$

$$\Rightarrow \lambda = \frac{-\gamma \pm \sqrt{\gamma^2 + 4\omega^2}}{2}$$

$$\lambda_1 = \frac{-\gamma + \sqrt{\gamma^2+4\omega^2}}{2} > 0$$

$$\lambda_2 = \frac{-\gamma - \sqrt{\gamma^2+4\omega^2}}{2} < 0$$

Eigenvalues are real & opposite signed.

$\Rightarrow$ SADDLE. (both of linear system and of the locally linear system).

$\lambda_1 \xi_1 = \xi_2$

$\frac{-\gamma - \gamma - \sqrt{\Delta}}{2} = \frac{-2\gamma - \sqrt{\Delta}}{2}$

Eigenvectors:

$\frac{-\gamma - \gamma + \sqrt{\Delta}}{2}$

$$\begin{bmatrix} -\lambda_1 & 1 \\ \omega^2 & -\gamma-\lambda_1 \end{bmatrix}\begin{bmatrix} \xi_1 \\ \xi_2 \end{bmatrix} = 0 \Rightarrow -\lambda_1\xi_1 + \xi_2 = 0 \Rightarrow \begin{bmatrix} \xi_1 \\ \xi_2 \end{bmatrix} = \begin{bmatrix} 1 \\ \lambda_1 \end{bmatrix}$$

$$\omega^2\xi_1 + (-\gamma-\lambda_1)\xi_2 = 0$$

$$-\gamma - \lambda_1 = \frac{-\gamma - \gamma \cdot \sqrt{\gamma^2+4\omega^2}}{2} = \frac{-\sqrt{\gamma^2+4\omega^2}}{2} = \frac{-\sqrt{\Delta}}{2}$$

$$\Rightarrow \omega^2\xi_1 = (-\gamma-\lambda_1)\xi_2 \Rightarrow \omega^2\xi_1 = \frac{-\sqrt{\Delta}}{2}\xi_2$$

$$\Rightarrow \begin{bmatrix} \xi_1 \\ \xi_2 \end{bmatrix} = \begin{bmatrix} -\gamma - \frac{\sqrt{\Delta}}{2} \\ \omega^2 \end{bmatrix}$$

$$\begin{bmatrix} -\lambda_2 & 1 \\ \omega^2 & -\gamma-\lambda_2 \end{bmatrix}\begin{bmatrix} \xi_1 \\ \xi_2 \end{bmatrix} = 0 \Rightarrow -\lambda_2\xi_1 + \xi_2 = 0 \Rightarrow \begin{bmatrix} \xi_1 \\ \xi_2 \end{bmatrix} = \begin{bmatrix} 1 \\ \lambda_2 \end{bmatrix}$$

$$\omega^2\xi_1 + (-\gamma-\lambda_2)\xi_2 = 0 \Rightarrow \begin{bmatrix} \xi_1 \\ \xi_2 \end{bmatrix} = \begin{bmatrix} -\gamma + \frac{\sqrt{\Delta}}{2} \\ \omega^2 \end{bmatrix}$$

Eigenvectors:

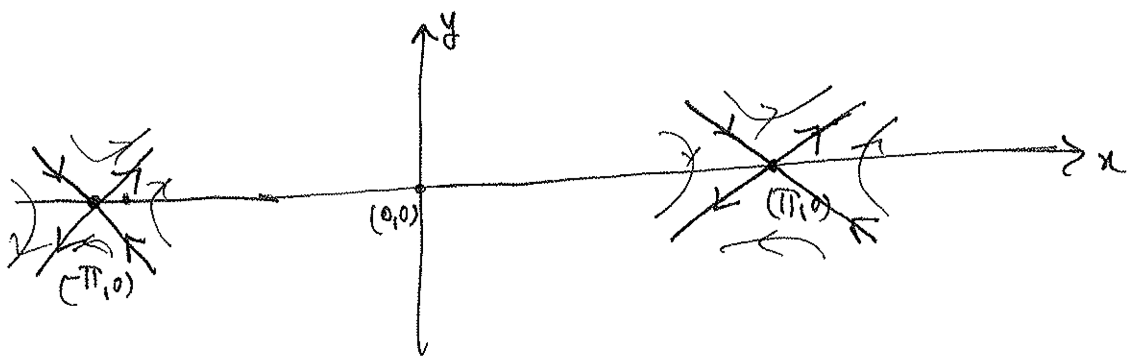
$\lambda_1 : \longrightarrow \begin{bmatrix} 1 \\ \lambda_1 \end{bmatrix}$ $\qquad \lambda_1 > 0$

$\qquad \lambda_2 < 0$

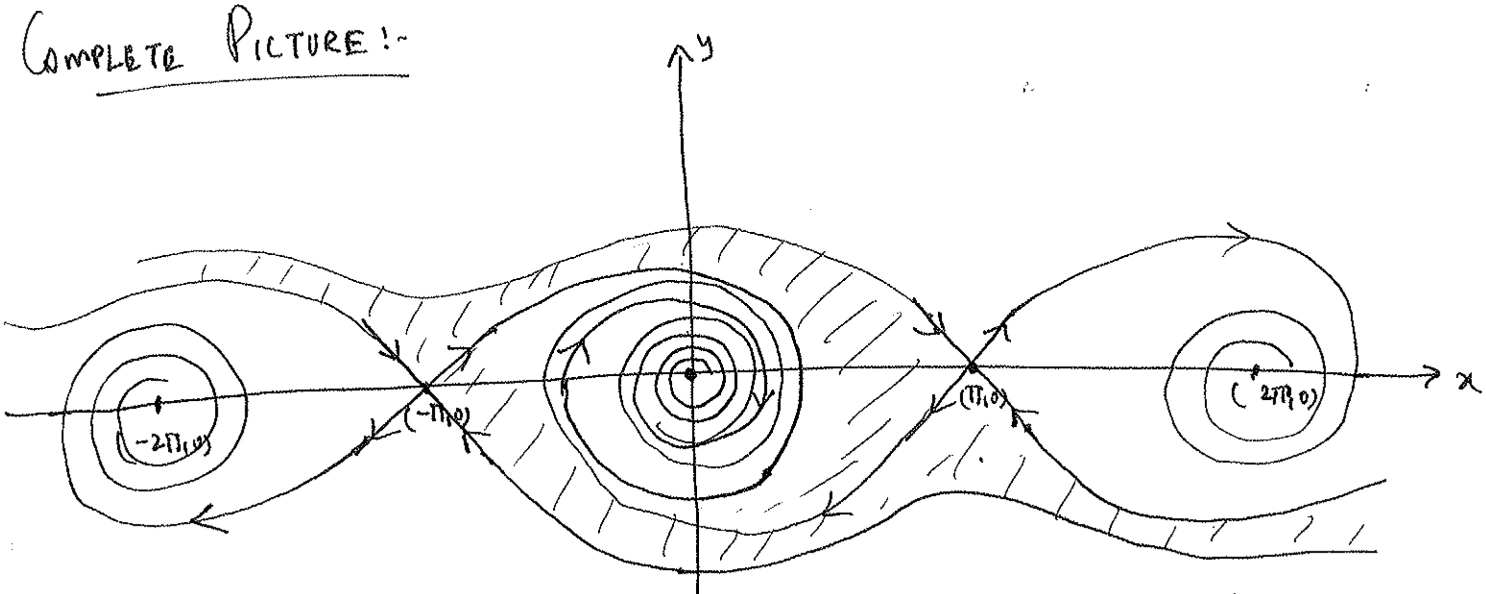
$\lambda_2 : \longrightarrow \begin{bmatrix} 1 \\ \lambda_2 \end{bmatrix}$

$$\therefore \begin{bmatrix} u \\ v \end{bmatrix} = c_1 \begin{bmatrix} 1 \\ \lambda_1 \end{bmatrix} e^{\lambda_1 t} + c_2 \begin{bmatrix} 1 \\ \lambda_2 \end{bmatrix} e^{\lambda_2 t}$$

Negative slope $\qquad$ positive slope



COMPLETE PICTURE :-



Origin is not globally asymptotic stable, but only locally stable.