Nonlinear Problems

General Procedure: Each problem has the form:

$$\left.\begin{aligned} x' &= F(x,y) \\ y' &= G(x,y) \end{aligned}\right\} \quad \vec{X}' = \vec{f}(\vec{X})$$

Step 1: find critical / equilibrium points that satisfy $F(x,y) = 0$ & $G(x,y) = 0$.

Step 2: If (x_0, y_0) is a critical point, then let

$$\left.\begin{aligned} u_1 &= x - x_0 \\ u_2 &= y - y_0 \end{aligned}\right\} \quad \begin{aligned} u_1' &= x' \\ u_2' &= y' \end{aligned}$$

Linearizing, we get

$$\begin{bmatrix} u_1' \\ u_2' \end{bmatrix} = \begin{bmatrix} F_x & F_y \\ G_x & G_y \end{bmatrix}_{\text{at } (x_0,y_0)} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$$

$$\Rightarrow \quad \vec{u}' = J\vec{u} \quad \text{where} \quad J = \begin{bmatrix} F_x & F_y \\ G_x & G_y \end{bmatrix}_{(x_0,y_0)}$$

(Jacobian).

Step 3: For each critical point, calculate eigenvalues of J and classify stability (saddle, spiral, center, etc.). Draw local trajectories near $(0,0)$.

Step 4: Since $x = u_1 + x_0$, $y = u_2 + y_0$, shift the local trajectories to (x_0, y_0). After doing this for all critical points, connect the trajectories to construct the global phase portrait.

Ex: $x' = x(y-4)$ $\quad$ x : Predator

$y' = y(4-x)$ $\quad$ y : Prey

$$F(x,y) = x(y-4)$$
$$G(x,y) = y(4-x)$$

(i) Critical Points: $F(x,y) = 0$ & $G(x,y) = 0$

$\Rightarrow x(y-4) = 0$ & $y(4-x) = 0$

$(0,0)$, $(4,4)$

$$J = \begin{bmatrix} F_x & F_y \\ G_x & G_y \end{bmatrix}_{(x_0,y_0)} = \begin{bmatrix} y_0 - 4 & x_0 \\ -y_0 & 4 - x_0 \end{bmatrix}$$

(ii) Locally linear system: $\vec{u}\,' = J_{(0,0)}\,\vec{u}$

near $(0,0)$:

$$\Rightarrow \vec{u}\,' = \begin{bmatrix} -4 & 0 \\ 0 & 4 \end{bmatrix} \vec{u}$$

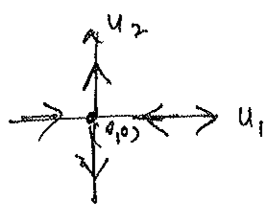
Eigenvalues: $\lambda_1 = 4$, $\lambda_2 = -4$ $\Rightarrow$ $(0,0)$ is a saddle point.

Eigenvectors: • $(J - \lambda_1 I)\vec{\xi} = 0$

$$\Rightarrow \begin{bmatrix} -8 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \xi_1 \\ \xi_2 \end{bmatrix} = \vec{0} \Rightarrow \vec{\xi} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

• $(J - \lambda_2 I)\vec{\xi} = 0$

$$\Rightarrow \begin{bmatrix} 0 & 0 \\ 0 & 8 \end{bmatrix} \begin{bmatrix} \xi_1 \\ \xi_2 \end{bmatrix} = \vec{0} \Rightarrow \vec{\xi} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$



Near $\underline{(4,4)}$: $\quad \vec{u}' = J_{(4,4)} \vec{u}$

$$\Rightarrow \vec{u}' = \begin{bmatrix} 0 & 4 \\ -4 & 0 \end{bmatrix} \vec{u}$$

Eigenvalues: $(-\lambda)(-\lambda) + 16 = 0 \Rightarrow \lambda^2 + 16 = 0 \Rightarrow \lambda_\pm = \pm 4i$: CENTER

Eigenvectors: $\left(\begin{bmatrix} 0 & 4 \\ -4 & 0 \end{bmatrix} - 4i \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right) \vec{\xi} = 0$

$$\Rightarrow \begin{bmatrix} -4i & 4 \\ -4 & -4i \end{bmatrix} \begin{bmatrix} \xi_1 \\ \xi_2 \end{bmatrix} = 0 \Rightarrow \begin{bmatrix} -i & 1 \\ -1 & -i \end{bmatrix} \begin{bmatrix} \xi_1 \\ \xi_2 \end{bmatrix} = 0$$

$$\Rightarrow i\xi_1 = \xi_2 \Rightarrow \vec{\xi} = \begin{bmatrix} 1 \\ i \end{bmatrix}$$

Solution: $\vec{u}^{(1)} = \begin{bmatrix} 1 \\ i \end{bmatrix} e^{i4t} = \left(\begin{bmatrix} 1 \\ 0 \end{bmatrix} + i \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right) (\cos 4t + i \sin 4t)$

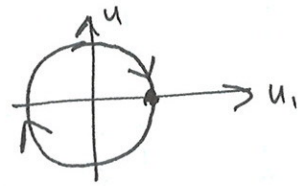
Real Part, $\vec{u}_R = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \cos(4t) - \begin{bmatrix} 0 \\ 1 \end{bmatrix} \sin 4t$

$$= \begin{bmatrix} \cos 4t \\ -\sin 4t \end{bmatrix} \longrightarrow \text{circle of radius } r = 4$$

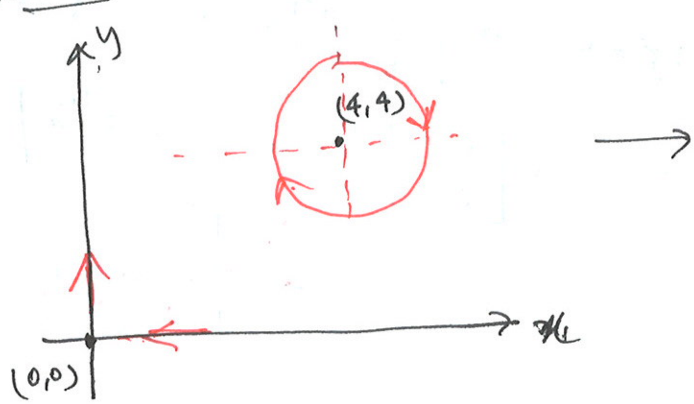
Direction: $u_1' = 4u_2$
$u_2' = -4u_1$

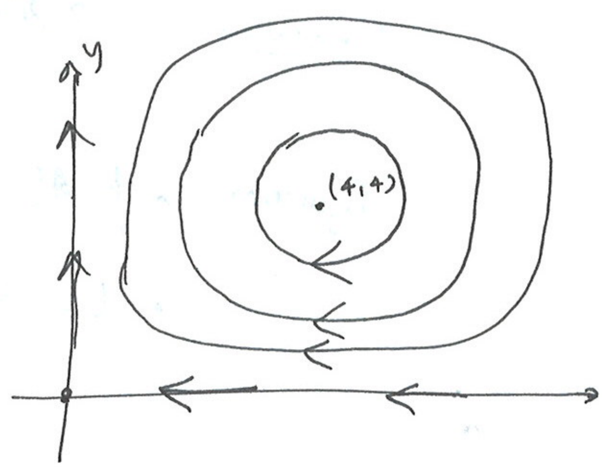
At $u_1 > 0$ & $u_2 = 0$, $u_2' = -4u_1 < 0$

$\Rightarrow$ clockwise orbit.



Global Phase Portrait:





Ex:

$$x' = x(2-2x-y) \quad , \qquad x>0, \; y>0$$
$$y' = y(2-2y-x)$$

$$F(x,y) = x(2-2x-y)$$
$$G(x,y) = y(2-2y-x)$$

(i) Critical Points:- $\quad F=0 \;\;\&\;\; G=0$

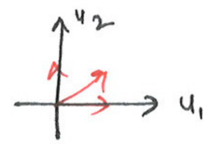
$$\Rightarrow (0,0),\; (1,0),\; (0,1),\; \left(\tfrac{2}{3}, \tfrac{2}{3}\right)$$

(ii) Jacobian $\quad J = \begin{bmatrix} 2-4x-y & -x \\ -y & 2-4y-x \end{bmatrix}$

$\rightarrow$ Near (0,0): $\quad \vec{u}' = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} \vec{u}$ $\quad$: Unstable node.

Eigenvalues : $\lambda_1 = \lambda_2 = 2$

Eigenvectors : $\vec{\xi}^{(1)} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, $\vec{\xi}^{(2)} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$



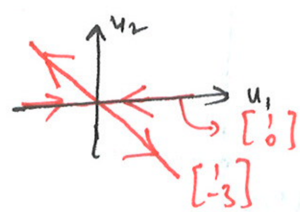
$\rightarrow$ Near (1,0) : $\quad \vec{u}' = \underbrace{\begin{bmatrix} -2 & -1 \\ 0 & 1 \end{bmatrix}}_{J} \vec{u}$

Eigenvalues: $(-2-\lambda)(1-\lambda) = 0 \rightarrow \lambda_1 = 1 \;\;\&\;\; \lambda_2 = -2$

$\Rightarrow (1,0)$ is a saddle.

Eigenvectors: • $(J-\lambda_1 I)\vec{\xi} = 0 \rightarrow \begin{bmatrix} -3 & -1 \\ 0 & 0 \end{bmatrix} \vec{\xi} = 0 \Rightarrow \vec{\xi}^{(1)} = \begin{bmatrix} 1 \\ -3 \end{bmatrix}$

• $(J-\lambda_2 I)\vec{\xi} = 0 \rightarrow \begin{bmatrix} 0 & 1 \\ 0 & 3 \end{bmatrix} \vec{\xi} = 0 \Rightarrow \vec{\xi}^{(2)} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$



→ Near $\underline{(0,1)}$: $\quad \vec{u}\,' = \begin{bmatrix} 1 & 0 \\ -1 & -2 \end{bmatrix} \vec{u}$

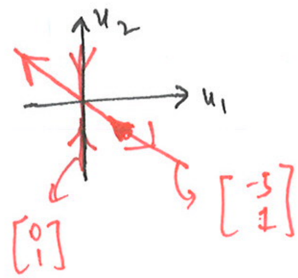
Eigenvalues: $\quad (1-\lambda)(-2-\lambda) = 0 \quad \Rightarrow \quad \lambda_1 = 1 \; , \; \lambda_2 = -2$

(SADDLE POINT)

Eigenvectors: • $(J - \lambda_1 I)\vec{\xi} = 0 \quad \Rightarrow \quad \begin{bmatrix} 0 & 0 \\ -1 & -3 \end{bmatrix} \vec{\xi} = 0$

$\rightarrow \quad \vec{\xi}^{(1)} = \begin{bmatrix} -3 \\ 1 \end{bmatrix}$

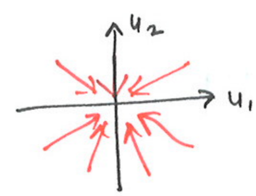
• $(J - \lambda_2 I)\vec{\xi} = 0 \quad \Rightarrow \quad \begin{bmatrix} 3 & 0 \\ -1 & 0 \end{bmatrix} \vec{\xi} = 0 \quad \Rightarrow \quad \vec{\xi}^{(2)} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$



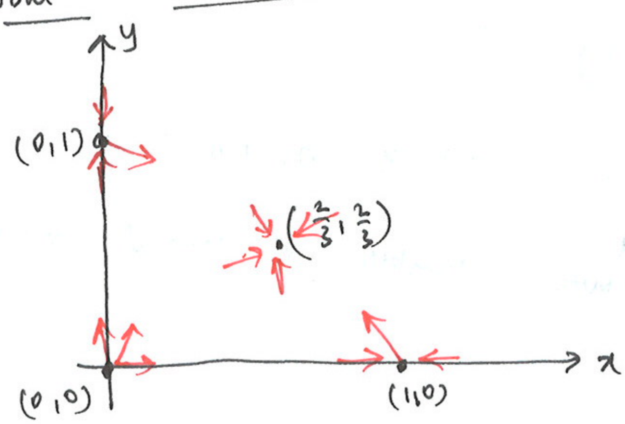
→ Near $\underline{\left(\frac{2}{3}, \frac{2}{3}\right)}$: $\quad \vec{u}\,' = \begin{bmatrix} -4/3 & -2/3 \\ -2/3 & -4/3 \end{bmatrix} \vec{u}$

Eigenvalues: $\quad \left(-\frac{4}{3} - \lambda\right)\left(-\frac{4}{3} - \lambda\right) - \frac{4}{9} = 0 \quad \Rightarrow \quad \lambda_1 = \frac{-2}{3} \; , \; \lambda_2 = -2$

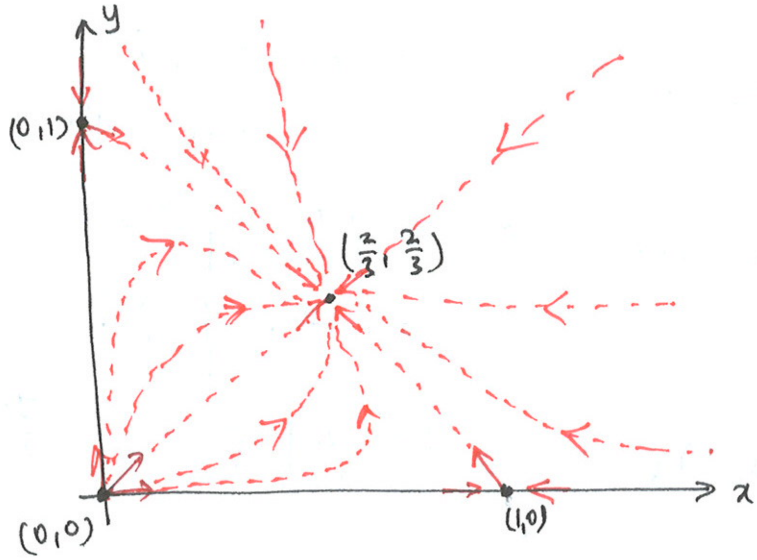
$\left(\frac{2}{3}, \frac{2}{3}\right)$ is a stable node.



Global Phase Portrait: $\quad$ Putting it all together.

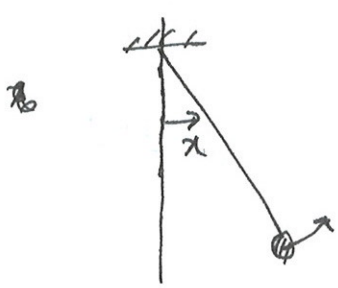


Joining the curves, we get



Its very clear that $(\frac{2}{3}, \frac{2}{3})$ is a global attractor.

Ex: Pendulum: Undamped case



$$\left.\begin{aligned} x' &= y \\ y' &= -\gamma y - \omega^2 \sin x \end{aligned}\right\} \text{General System}$$

γ : damping.

$x \to$ angle

$y \to$ angular velocity.

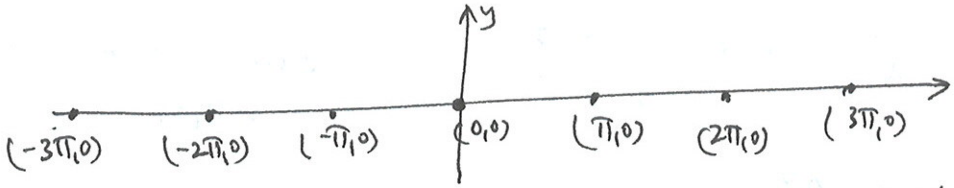
~~Und~~

Simpler case: Damping, $\gamma = 0$

$$\Rightarrow \quad \begin{aligned} x' &= y \\ y' &= -\sin x \end{aligned} \qquad (\text{assuming } \omega = 1)$$

Critical Points: $y = 0$ & $\sin x = 0 \Rightarrow x = 0, \pm\pi, \pm 2\pi, \ldots$

The system has infinite number of critical points.



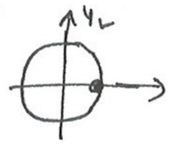
Let us analyse the system near three critical points:

(x_0, y_0) : $(-\pi, 0)$, $(0,0)$, $(\pi, 0)$.

Jacobian: $J = \begin{bmatrix} 0 & 1 \\ -\cos x & 0 \end{bmatrix}_{(x_0, y_0)}$

→ Near $(0,0)$: $\vec{u}' = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \vec{u}$

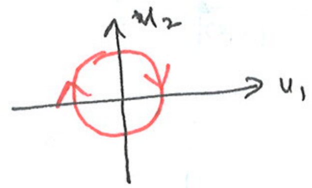
Eigenvalues: $(-\lambda)(-\lambda) + 1 = 0 \Rightarrow \lambda^2 + 1 = 0 \Rightarrow \lambda_{\pm} = \pm i$



$\Rightarrow (0,0)$ is a center.

Direction: $\left.\begin{aligned} u_1' &= u_2 \\ u_2' &= -u_1 \end{aligned}\right\}$ At $u_1 > 0$ & $u_2 = 0$, $u_2' = -u_1 < 0 \Rightarrow u_2$ decreases

$\Rightarrow$ The orbit is clockwise.



→ Near $(\pi, 0)$: $\vec{u}' = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \vec{u}$

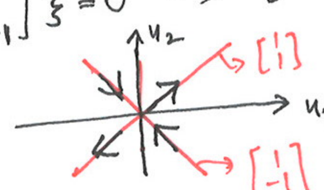
Eigenvalues: $(-\lambda)(-\lambda) - 1 = 0 \Rightarrow \lambda^2 = 1 \Rightarrow \cancel{\lambda_1 = 1, \lambda_2 = 1}$

$\Rightarrow \lambda^2 - 1 = 0 \Rightarrow \lambda = \pm 1$

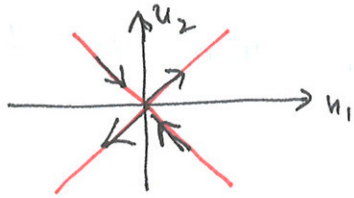
$\lambda_1 = -1, \quad \lambda_2 = 1$

Eigenvectors: • $(J - \lambda_1 I)\vec{\xi} = 0 \Rightarrow \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \vec{\xi} = 0 \Rightarrow \vec{\xi}^{(1)} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$

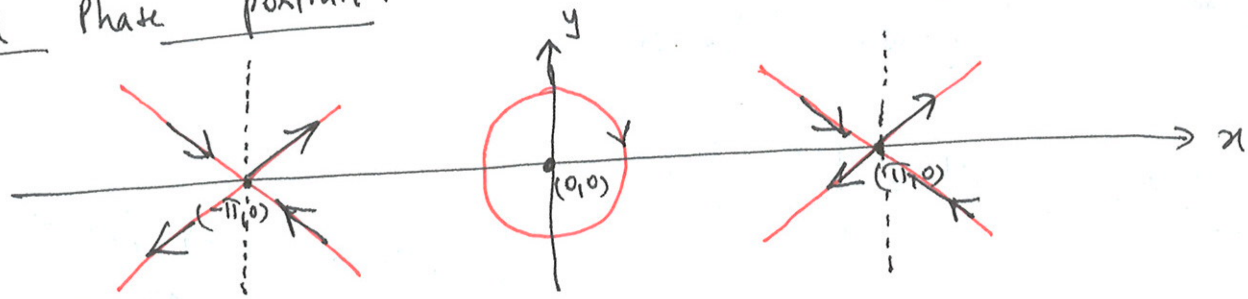
• $(J - \lambda_2 I)\vec{\xi} = 0 \Rightarrow \begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix} \vec{\xi} = 0 \Rightarrow \vec{\xi}^{(2)} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$



Near $(-\pi, 0)$: $\vec{u}' = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \vec{u}$: Same system as near $(\pi, 0)$



Global Phase portrait :



Connecting the trajectories :