METHOD OF INTEGRATING FACTORS
(FIRST ORDER LINEAR DIFFERENTIAL EQUATIONS)

① Standard form :-
$$y' + p(x)\,y = q(x) \quad \text{—— ①}$$
$$y(0) = y_0 \quad \text{—— ②}$$

② Multiply the equation by an unknown function $\phi(x)$.

This gives
$$\phi y' + p\phi y = q\phi \quad \text{—— ③}$$

If we can write equation ③ in the form
$$(\phi y)' = q\phi,$$
then we can integrate on both sides to obtain $y(x)$.

Since $(\phi y)' = \phi y' + \phi' y$, we require
$$\boxed{\phi' = p\phi} \quad \text{—— ④.}$$
$$\Rightarrow \quad \boxed{\phi = e^{\int_0^x p(s)\,ds}}$$
where s is a dummy variable for integration.

(LOWER LIMIT FROM INTIAL CONDITION)

We don't bother with the constant of integration since it is sufficient to find just one integrating factor.

③ Equation ③ becomes
$$(\phi y)' = q\phi \quad \text{—— ⑤}$$

4 Integrate for y:

$$\int (\phi y)' \, dx = \int_0^x \phi g \, dx + \text{constant}$$

$$\Rightarrow \quad \phi y = \int_0^x \phi g \, dx + C$$

~~NOTE! $\phi(x)$ can vanish if $p(x)$ is discontinuous. If that happens, $y(x)$ will cease to exist.~~

Notice $\phi(0) = 1 \Rightarrow C = y_0$ — 6

$$\Rightarrow \quad \boxed{y(x) = \frac{1}{\phi(x)}\left[\int_0^x \phi(s)\, g(s)\, ds + y_0\right]} \quad \text{— 7}$$

↳ If $\phi(x) \to 0$; $y(x)$ ceases to exist.

Ex: Solve for $y(x)$ in

$$x y' + 2y = x^2 - x + 1$$

$$y(1) = \frac{1}{2}$$

Step 1: Write in std. form

$$y' + \left(\frac{2}{x}\right) y = x - 1 + \frac{1}{x}$$

$$p(x) = \frac{2}{x}$$

$$\Rightarrow \frac{\phi'}{\phi} = \frac{2}{x} \Rightarrow \phi = e^{\int^x \frac{2}{s} ds} = e^{2\log x} = x^2$$

(log : Natural logarithm)

Step 3: Multiply by ϕ, on both sides

$$\phi y' + \left(\frac{2}{x}\right)\phi y = \left(x - 1 + \frac{1}{x}\right)\phi$$

$$\Rightarrow \quad x^2 y' + 2xy = x^3 - x^2 + x$$

from the product rule, the left hand side becomes $(x^2 y)'$

$$\Rightarrow \quad (x^2 y)' = x^3 - x^2 + x$$

Step 4: Integrating both sides:

$$x^2 y = \frac{x^4}{4} - \frac{x^3}{3} + \frac{x^2}{2} + C$$

$$\Rightarrow \quad \boxed{y = \frac{x^2}{4} - \frac{x}{3} + \frac{1}{2} + \frac{C}{x^2}} \longrightarrow \text{GENERAL SOLUTION}$$

Step 5: FIND C:

$$y(1) = \frac{1}{2}$$

$$\Rightarrow \quad \frac{1}{2} = \frac{1}{4} - \frac{1}{3} + \frac{1}{2} + C \quad \Rightarrow \quad C = \frac{1}{12}$$

$$\therefore \quad \boxed{y = \frac{x^2}{4} - \frac{x}{3} + \frac{1}{2} + \frac{1}{12x^2}} \longrightarrow \text{PARTICULAR SOLUTION}$$

FURTHER COMMENTS ON THE SOLUTION:-

Since $\phi(x) = x^2$, $\phi(x) = 0$ at $x = 0$.

from equation (7) on page (2), when $\phi(x) \to 0$, $y(x) \to \infty$.

That is exactly what happens here too.

At $x = 0$; $y = \infty$. Therefore solution does not exist at $x = 0$.

This is something that cannot be predicted easily by looking at the differential equation.