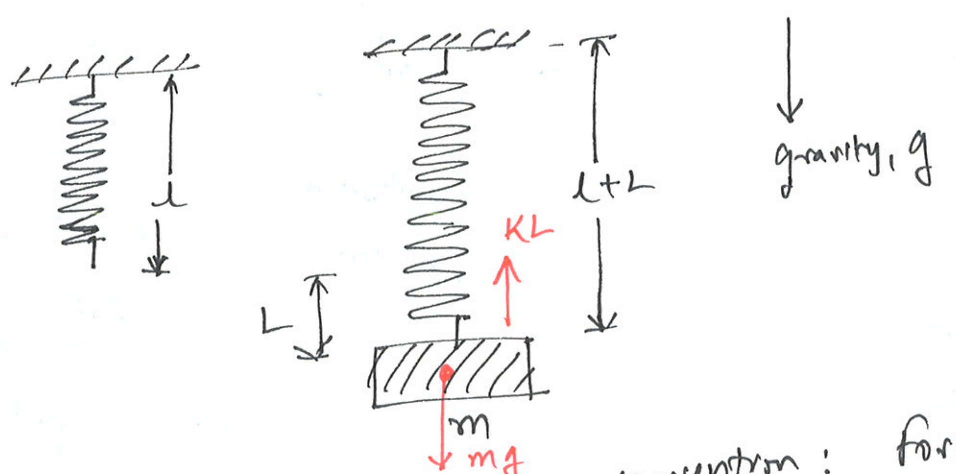
SECTION 3.7 Mechanical Vibrations

Second order linear equations with constant coefficients serve as useful mathematical models to describe mechanical oscillations.

SPRING-MASS SYSTEM: Consider a mass 'm' hanging on a spring of length 'l' (unstretched length). Due to the weight of the object, the spring stretched by a certain length 'L'.



We use the following sign convention: forces pointing downwards are positive & forces upwards are negative.

There are two forces on the mass:

(i) Weight, mg, which stretches the spring.

(ii) Spring force making the spring recoil upwards.

According to Hooke's Law, spring force

$F_s = -KL$. $K > 0$ is called the spring constant.

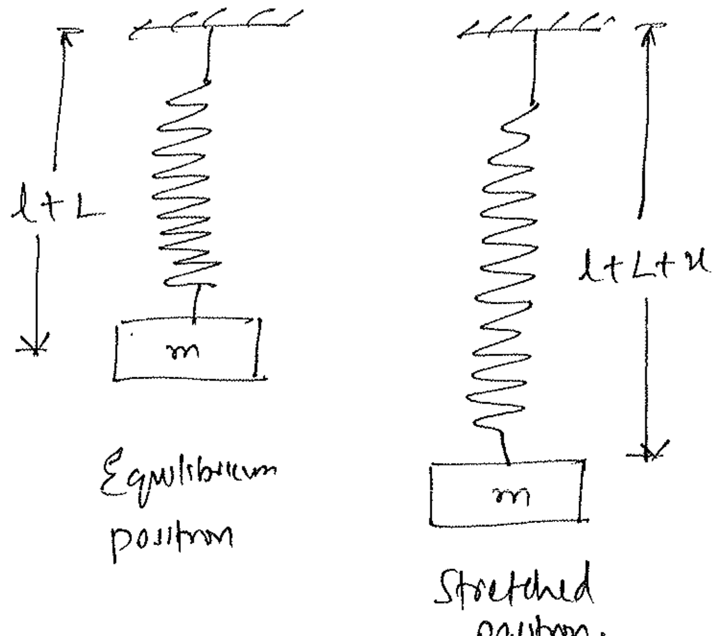
At equilibrium, the forces are balanced.

$$\Rightarrow \quad mg - KL = 0$$

$$\Rightarrow \quad L = \frac{mg}{K}$$: Equilibrium extension of the spring.

Now, let us stretch the spring beyond the equilibrium point.

Let $u(t)$ be the extension from the equilibrium point.



Again $u(t)$ is positive in the downward direction.

If we pull the mass by $u(t)$ & let it go, will we see an oscillation?

From Newton's law,

$$\text{force} = m \times \text{acceleration}$$
$$= m \times \frac{d^2u}{dt^2}$$

<u>Force</u>: There are again two forces:

(i) Weight, mg

(ii) Spring force, which is now $-K(L+u)$ (upwards)

$$\Rightarrow \quad \underbrace{mg}_{\substack{\text{Weight,}\\ mg}} - \underbrace{K(L+u)}_{\substack{\text{Spring force,}\\ f_s}} = m\frac{d^2u}{dt^2}$$

Spring force: $f_s = -K(L+u)$; $\boxed{K = \text{SPRING CONSTANT}}$

- If $L+u<0$, the spring is compressed by a distance $|L+u|$, and the force is directed dowards, and is given by $f_s = K|L+u| = -K(L+u)$
- If $L+u>0$, the spring is stretched by a distance $|L+u|$. But the force is now directed upwards. In this case $f_s = -K|L+u| = -K(L+u)$

$\Rightarrow$ We have the same formula for the spring force irrespective of the direction of stretched.

Now, we can consider a more sophisticated model where there could be a resistive force. This resistive force dampens the motion of the mass

DAMPING FORCE:- $f_d = -\gamma \dfrac{du}{dt}$: Viscous damping.

$$\boxed{\gamma = \text{DAMPING CONSTANT}}$$

Such a damping force is a good model to describe the resistive force of a DASHPOT, the device which prevents the door from slamming.

If $\frac{du}{dt} > 0$, then u is increasing $\Rightarrow$ mass is moving downwards

$\Rightarrow F_d = -\gamma u'(t)$ is directed upwards.

If $\frac{du}{dt} < 0$, then u is decreasing. $\Rightarrow$ the mass is moving upwards

$\Rightarrow F_d = -\gamma u'(t)$ is directed downwards.

Including the damping term, the modified spring-mass-damper model becomes:

$$m u''(t) = mg - K(L+u) - \gamma u'(t).$$

$$\Rightarrow \quad m u''(t) + \gamma u'(t) + Ku = (mg - KL)$$

But note that $mg = KL \Rightarrow mg - KL = 0$

$$\therefore \quad \boxed{m u''(t) + \gamma u'(t) + Ku = 0}$$

- We can further generalize the model to include the effect of an external force, $F(t)$. Typically, this external force could be due to an ~~constant~~ external forcing on the mass 'm'.

With an external force, $F(t)$, the model becomes

$$m u''(t) + \gamma u'(t) + K u(t) = F(t).$$

- At $t = 0$, we can specify the initial position and velocity of the mass. With these initial conditions, the final model for a SPRING-MASS-DAMPER becomes

$$\boxed{\begin{aligned} &m u'' + \gamma u' + K u = F(t), \\ &u(0) = u_0 \quad ; \quad u'(0) = u_0 \quad ; \end{aligned}}$$

m, γ, K are positive constants.

↳ 2nd order linear equation with constant coefficients.

- To study the above model, we first consider simple sub-cases.

CASE - I : UNDAMPED FREE VIBRATIONS

UNDAMPED $\downarrow$ $\gamma = 0$ (no damping)

FREE $\downarrow$ No forcing $\Rightarrow F(t) = 0$

The model becomes

$$\boxed{m u'' + K u = 0}, \quad K > 0$$

Characteristic equation : $m r^2 + K = 0$

$$\Rightarrow r = \pm i \sqrt{\frac{K}{m}} = \pm i \omega_0$$

where $\omega_0 = \sqrt{\frac{K}{m}}$

General Solution : $u(t) = A\cos(\omega_0 t) + B\sin(\omega_0 t)$ — (*)

We can rewrite this result in the form :

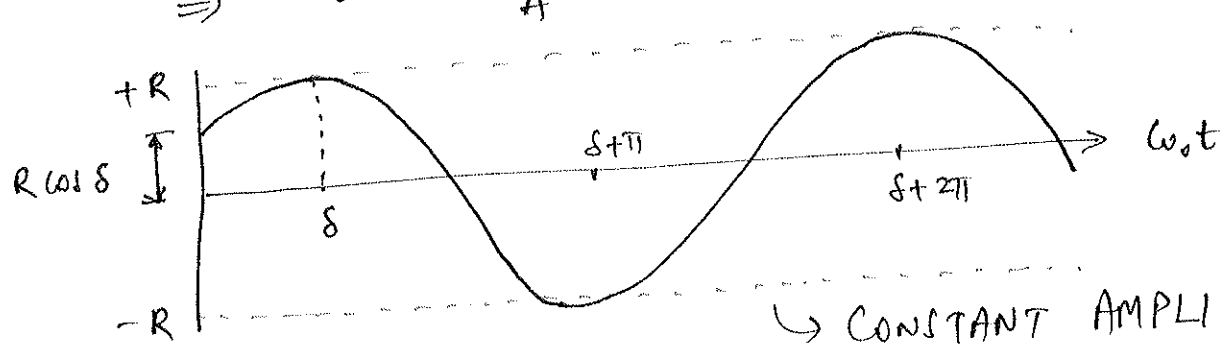
R = Amplitude, δ : Phase

$$u(t) = R\cos(\omega_0 t - \delta) \;; \qquad \text{— } (**)$$
$$= R\left[\cos\delta\cos(\omega_0 t) + \sin\delta\sin(\omega_0 t)\right]$$

Comparing the two equations ((*) & (**)), we should have

$R\cos\delta = A$; $R\sin\delta = B$

$$\Rightarrow \tan\delta = \frac{B}{A}, \quad R = \sqrt{A^2 + B^2}$$



↳ CONSTANT AMPLITUDE

- Because of the absence of damping, the amplitude of oscillation remains constant. Moreover, the absence of external force results in perpetual oscillations.

- Phase δ relates the relative initial position of the mass vis-a-vis maximum amplitude of the oscillation.

- TIME - PERIOD :- $T = \dfrac{2\pi}{\omega_0} = 2\pi\sqrt{\dfrac{m}{K}}$

  ω_0 is also called the circular frequency.

→ If m increases, T increases ⇒ heavier masses oscillate slower.

→ If $\underbrace{K \text{ increases}}_{\text{stiffer spring}}$, T decreases ⇒ stiffer springs oscillate faster.

CASE - II :- DAMPED $\underbrace{\text{FREE}}_{\text{No forcing}}$ VIBRATIONS ⇒ $F(t) = 0$

The model becomes

$$\boxed{mu'' + \gamma u' + Ku = 0}, \quad m, \gamma, K > 0$$

Let $y = e^{rt}$

characteristic equation:- $m r^2 + \gamma r + K = 0$

$$\Rightarrow r_1, r_2 = \frac{-\gamma \pm \sqrt{\gamma^2 - 4Km}}{2m}$$

$$= \frac{-\gamma \pm \gamma\sqrt{1 - \frac{4Km}{\gamma^2}}}{2m}$$

$$= \frac{\gamma}{2m}\left[-1 \pm \sqrt{1 - \frac{4Km}{\gamma^2}}\right]$$

- If $\gamma^2 - 4Km > 0$, r_1 & r_2 are real and distinct.
- If $\gamma^2 - 4Km < 0$, r_1 & r_2 are complex conjugates.

Three cases:

(i) $\gamma^2 - 4Km > 0 \Rightarrow u(t) = Ae^{r_1 t} + Be^{r_2 t}$; $r_1 < 0$, $r_2 < 0$

(ii) $\gamma^2 - 4Km < 0 \Rightarrow u = e^{-\frac{\gamma t}{2m}}\left[A\cos\mu t + B\sin\mu t\right]$

with $\mu = \frac{(4Km - \gamma^2)^{1/2}}{2m}$

(iii) $\gamma^2 - 4Km = 0 \Rightarrow u = (A + Bt)e^{-\gamma t/2m}$.

Note that in all the three cases, $\lim_{t\to\infty} u(t) \Rightarrow 0$ for any A and B. This is because of the e^{rt} term with $r < 0$.

What does this physically mean? In the second and third case, $e^{-\frac{\gamma t}{2m}} \to 0$ as $t \to \infty$. This causes

⑤

$u(t) \to 0$ as $t \to \infty$. The presence of damping, γ, causes the oscillations to "die" down eventually vanishing in the limit $t \to \infty$.

In summary, No damping $\Rightarrow$ Steady Oscillations

Finite damping $\Rightarrow$ decaying or damping oscillations.

Let's examine the case with complex conjugate roots:

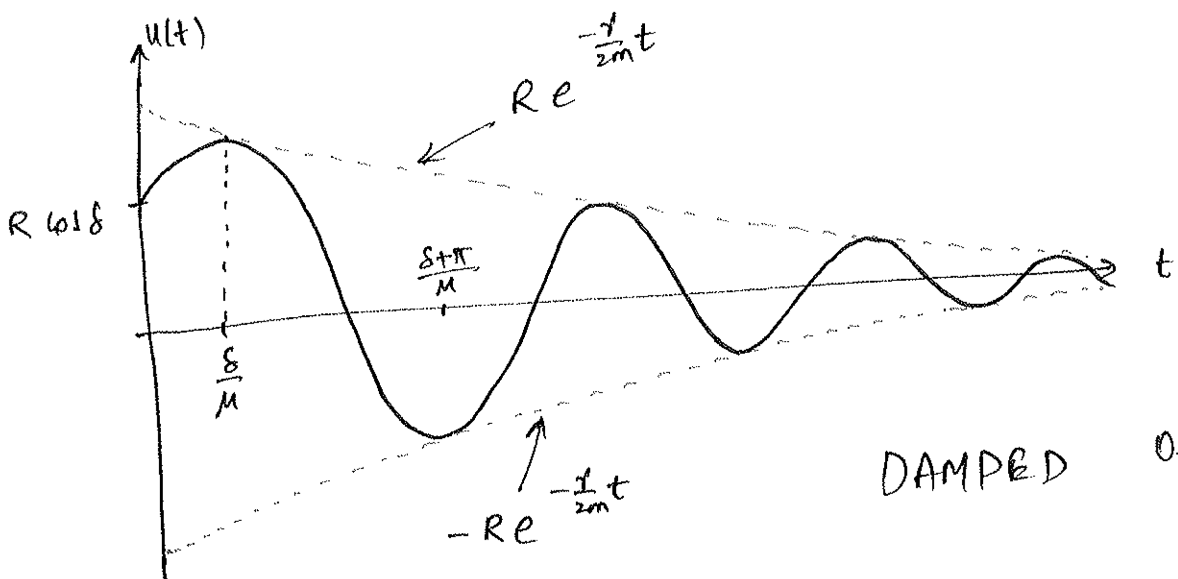
$$\gamma^2 - 4km < 0 \Rightarrow u(t) = e^{-\frac{\gamma t}{2m}} \left[A\cos(\mu t) + B\sin(\mu t) \right]$$

$$= R e^{-\frac{\gamma}{2m}t} \cos(\mu t - \delta)$$

where $A = R\cos\delta$,

$B = R\sin\delta$

and $\mu = \dfrac{\sqrt{4km - \gamma^2}}{2m}$.



DAMPED OSCILLATIONS

- Motion not periodic, but μ determines the frequency with which the mass oscillates. μ is called quasifrequency.

- $T_d = \frac{2\pi}{\mu}$: Quasi period.

Compare T_d with T : What happens with damping to the quasi period?

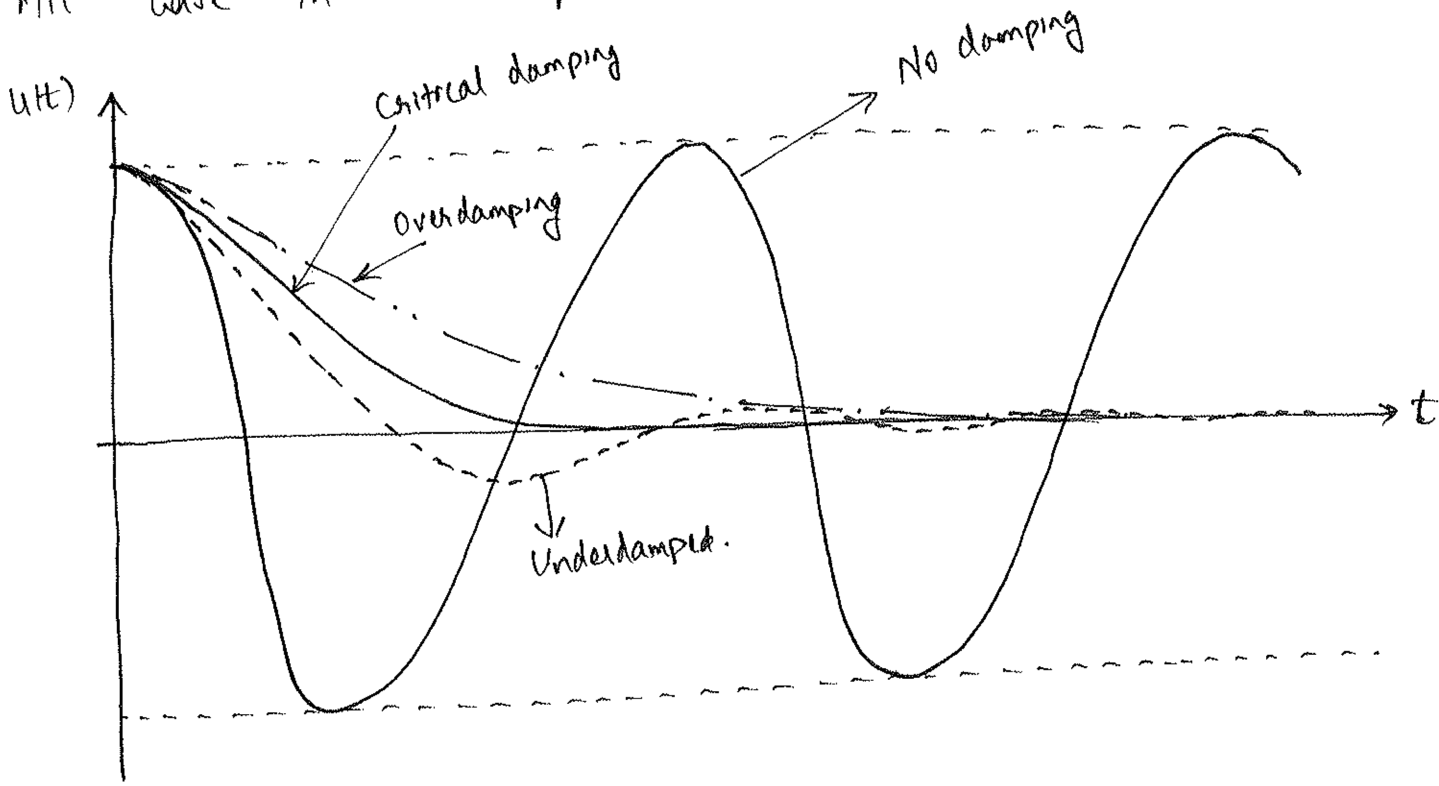
The nature of the solution changes at $\gamma^2 = 4km$.

- If $\gamma^2 = 4km \rightarrow$ CRITICAL DAMPING
  $\Rightarrow$ No oscillations, $\lim_{t\to\infty} u(t) = 0$.
  $\rightarrow$ Rapid decay of oscillations

- If $\gamma^2 > 4km \rightarrow$ OVERDAMPED
  $\Rightarrow$ No oscillations
  $\rightarrow$ Takes longer for $u(t) \to 0$ than in the critically damped case.

- If $\gamma^2 < 4km \rightarrow$ Underdamped.
  $\Rightarrow$ Oscillations are present.
  $\rightarrow$ Solution decays to equilibrium, but overshoots. It then recoils back. This continues to happen till the oscillations die down.

All case in one plot:



FORCED OSCILLATIONS

$$mu'' + \gamma u' + Ku = F(t)$$

We now study cases with $F(t)$ being an oscillatory function. Suppose $F(t) = F_0 \cos(\omega t)$, with $F_0 > 0$, $\omega > 0$.
↳ nonhomogeneous term.

Model becomes $mu'' + \gamma u' + Ku = F_0 \cos(\omega t)$

Case I: Forced Vibrations with Damping : $\gamma \neq 0$.

Ex: Let's start with a simple example.

Let $m = \gamma = 1$; $K = 2.5$, $F_0 = 1$; $\omega = 1$;

Initial conditions: $u(0) = 1$; $u'(0) = 2$

Equation becomes $u'' + u' + 2.5u = \cos t$

We can solve for $u(t)$ using the method of undetermined coefficients.

Homogeneous Solution:- $u'' + u' + 2.5u = 0$

$$\Rightarrow r^2 + r + 2.5 = 0$$

$$\Rightarrow \lambda = \frac{-1 \pm \sqrt{1 - 4 \times 2.5}}{2} = \frac{-1 \pm \sqrt{-9}}{2} = \frac{-1 \pm 3i}{2}$$

$$\Rightarrow u(t) = e^{-\frac{t}{2}}\left[A\cos\left(\frac{3t}{2}\right) + B\sin\left(\frac{3t}{2}\right)\right]$$

Particular Solution:-

Let $u_p(t) = C\cos t + D\sin t$

$u_p' = -C\sin t + D\cos t$

$u_p'' = -C\cos t - D\sin t$

$$\Rightarrow \left[-C\cos t - D\sin t\right] + \left[-C\sin t + D\cos t\right] + 2.5\left[C\cos t + D\sin t\right] = \cos t$$

$\Rightarrow \quad -C + D + 2.5C = 1 \quad \Rightarrow \quad 1.5C + D = 1$

and $\quad -D - C + 2.5D = 0 \quad \Rightarrow \quad 1.5D - C = 0$

$\Rightarrow \quad C = 1.5D$

$\Rightarrow \quad 1.5 \times 1.5D + D = 1$

$\Rightarrow \quad 3.25D = 1$

$\Rightarrow \quad D = \frac{1}{3.25}$

$\therefore \quad C = \frac{1.5}{3.25}$

$$\therefore \quad u_p(t) = \frac{1.5}{3.25}\cos t + \frac{1}{3.25}\sin t$$

$$\therefore \quad u(t) = e^{-t/2}\left[A\cos\left(\frac{3t}{2}\right) + B\sin\left(\frac{3t}{2}\right)\right] + \left(\frac{1.5}{3.25}\cos t + \frac{1}{3.25}\sin t\right)$$

Now $u(0) = 1 \Rightarrow 1 = A + \frac{1.5}{3.25} \Rightarrow A = \frac{1.75}{3.25}$

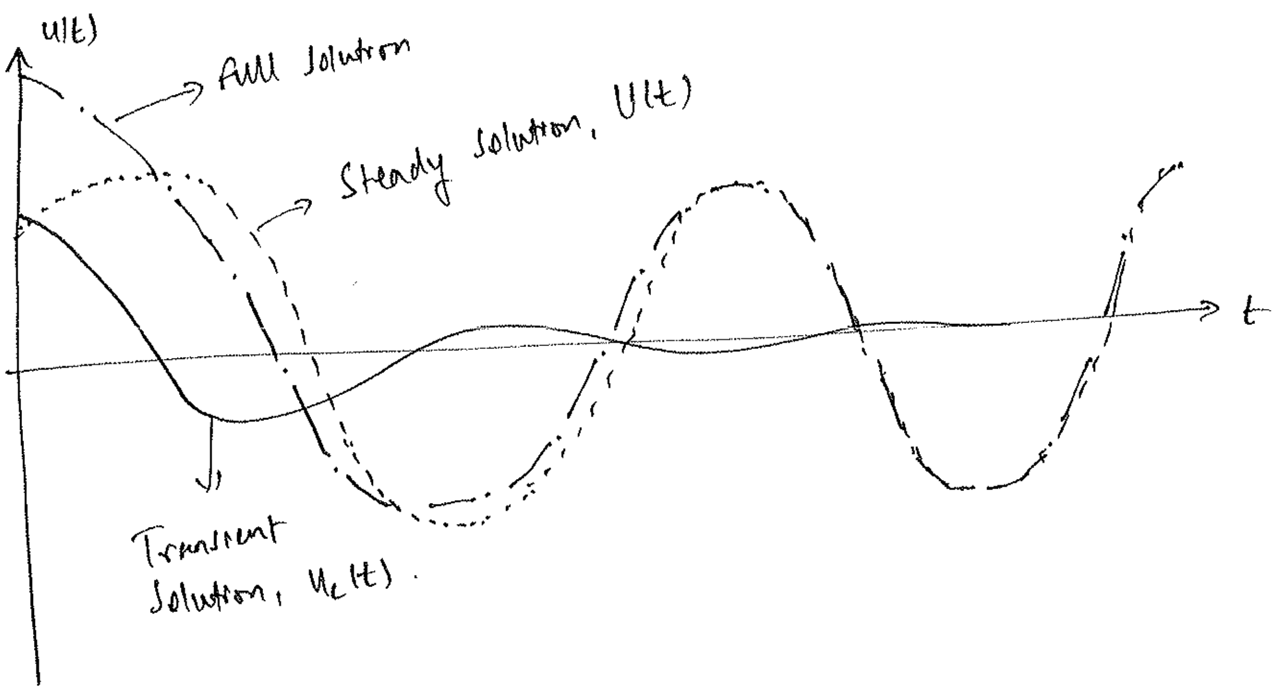
Similarly, $u'(0) = 0$ gives : $3B = A - \frac{2}{3.25} \Rightarrow B = \frac{-0.25}{9.75}$

$$\therefore\ u(t) = \underbrace{e^{-t/2}\left[\frac{1.75}{3.25}\cos\left(\frac{3t}{2}\right) - \frac{0.25}{9.75}\sin\left(\frac{3t}{2}\right)\right]}_{\text{Transient Solution}} + \underbrace{\left(\frac{1.5}{3.25}\cos t + \frac{1}{3.25}\sin t\right)}_{\text{Steady Solution}}$$

Transient Solution ⇓ Goes to zero as $t \to \infty$.

Steady Solution ⇓ Also called forced response.

$$= \quad u_c(t) \quad + \quad U(t)$$



Forced Vibrations with Damping: General Case

- $$m u''(t) + \gamma u'(t) + k u(t) = F_0 \cos(\omega t)$$

  where $m, \gamma, k, F_0, \omega$ are positive constants.

- F_0 represents the amplitude of the forcing & ω represents the frequency of the external forcing.

- The solution of the above equation can be obtained using method of undetermined coefficients. The general solution must have the form

$$u(t) = \underbrace{c_1 u_1(t) + c_2 u_2(t)}_{u_c(t)} + \underbrace{A\cos(\omega t) + B\sin(\omega t)}_{U(t)}$$

$$= u_c(t) + U(t)$$

- $u_1(t)$ & $u_2(t)$ are solutions of the homogeneous equation. Since m, γ, k are positive constants, the two roots of the characteristic equation, r_1 & r_2, are either negative or complex conjugates with a negative real part. This means that $u_1(t)$ & $u_2(t)$ decay with time as $t \to \infty$.

$$\therefore \lim_{t\to\infty} u_c(t) = \lim_{t\to\infty} c_1 u_1(t) + c_2 u_2(t) = 0$$

- $u_c(t)$ is called the <u>transient</u> solution because it survives only for a short time.

- $U(t)$ does not die down, but actually persists indefinitely. $U(t) = A\cos(\omega t) + B\sin(\omega t)$ represents steady oscillations and is the result of a response to the external forcing. $U(t)$ is therefore called the <u>steady</u> solution of the <u>forced</u> <u>response</u>.

We can rewrite $U(t)$ as

$$U(t) = R\cos(\omega t - \delta)$$

Expanding $\cos(\omega t - \delta)$, we get

$$U(t) = \underbrace{R\cos\delta}_{A}\cos(\omega t) + \underbrace{R\sin\delta}_{B}\sin(\omega t)$$

$$\therefore\quad R = \sqrt{A^2 + B^2}$$

$$\delta = \tan^{-1}\left(\frac{B}{A}\right)$$

After some tedious algebra, we can show that

$$R = \frac{F_0}{\sqrt{m^2(\omega_0^2 - \omega^2)^2 + \gamma^2\omega^2}} \quad ; \quad \cos\delta = \frac{m(\omega_0^2 - \omega^2)}{\sqrt{m^2(\omega_0^2 - \omega^2) + \gamma^2\omega^2}}$$

and $\sin\delta = \dfrac{\gamma\omega}{\sqrt{m^2(\omega_0^2-\omega^2)^2+\gamma^2\omega^2}}$

where $\omega_0^2 = \dfrac{K}{m}$: Natural frequency.

Resonance: Since R is a function of forcing frequency ω, we first evaluate the limiting cases.

When $\omega \approx 0$ (small frequency ~~oscillation~~ forcing),

$$R \approx \frac{F_0}{m\omega_0^2} \approx \frac{F_0}{K}$$: finite amplitude oscillation

When $\omega \to \infty$ (high frequency forcing),

$R \to 0$ (low amplitude oscillation).

It is reasonable to expect that $R(\omega)$ has a maximum value for some intermediate value of ω.

Using $\dfrac{\partial R}{\partial \omega} = 0$, we get $\omega_{max} = \sqrt{\omega_0^2 - \dfrac{\gamma^2}{2m^2}}$

↳ frequency at which amplitude R reaches a maximum.

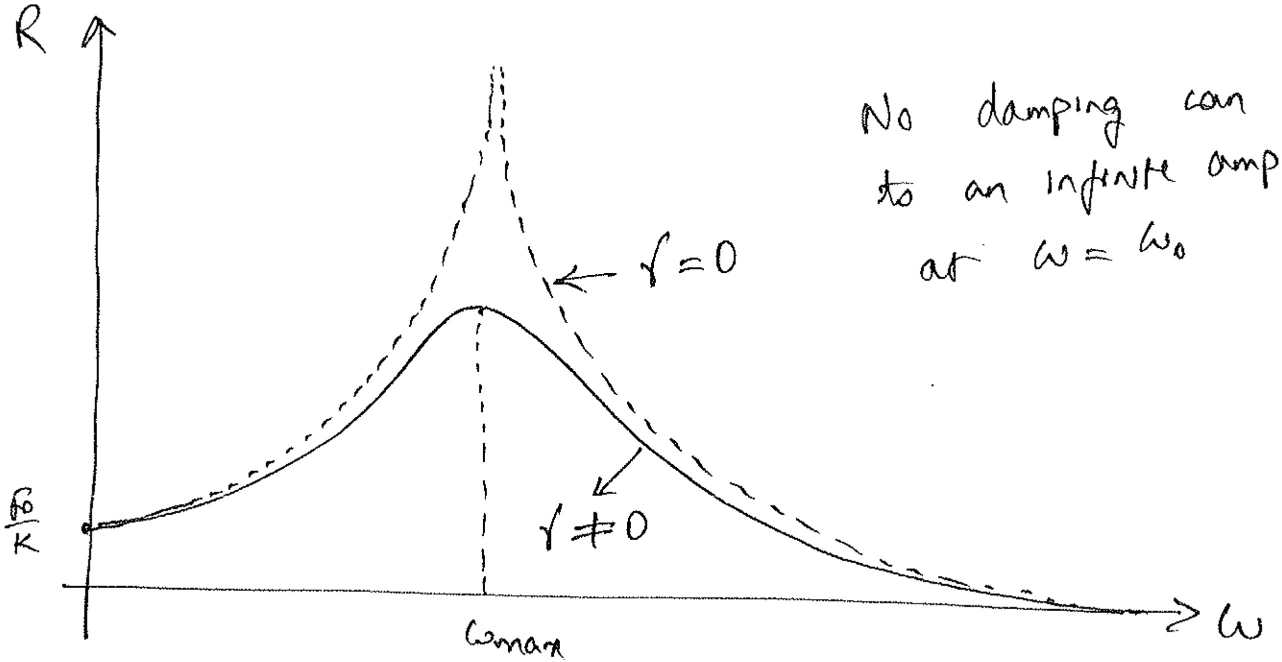
Using $\omega = \omega_{max}$, we get

$$R_{max} = \frac{F_0}{\gamma\omega_0\sqrt{1-\frac{\gamma^2}{4mK}}}.$$

Notice that $R_{max} \to \infty$ as $\underbrace{\gamma \to 0}_{\text{small damping}}$

$\Rightarrow$ Small damping can lead to large oscillations when $\omega \to \omega_{max}$ ($\approx \omega_0$).

This phenomenon is called RESONANCE



No damping can lead to an infinite amplitude at $\omega = \omega_0$