**PROBLEM 1:** (10 Points). Solve the initial value problem

$$y'' + 2y' + 2y = 0\,, \quad y(\pi/4) = 2\,, \quad y'(\pi/4) = -2\,.$$

**PROBLEM 2:** (15 Points). The problem

$$x^2 y'' + 3xy' + y = 0$$

has a solution of the form $y = x^p$ for some value of $p$. Find $p$ and then determine the general solution using reduction of order.

**PROBLEM 3:** (25 Points). Find the solution to

$$y'' + py' + y = 1 + e^{-x}\,, \qquad y(0) = y'(0) = 0$$

Account for all values of $p$ with $p$ real and $p \geq 0$ (There are several cases to consider). Calculate $\lim_{x\to\infty} y(x)$ in each case. Draw a plot of $y$ versus $x$ when $p = 0$.

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**PROBLEM 1:** (15 Points). Find the general solution to

$$y' - \frac{p}{x}y = x\,, \qquad \text{for } \; x \geq 0\,.$$

Here $p$ is a constant. For what values of $p$ is the solution undefined at $x = 0$?

**PROBLEM 2:** (15 Points). Solve the initial value problem

$$\frac{dy}{dx} = \frac{x(y-1)^2}{x-1}\,, \quad y(2) = 2\,.$$

What is the solution when the initial condition is changed to $y(2) = 1$?

**PROBLEM 4:** (10 Points). Consider the differential equation

$$\frac{dy}{dt} = -y(y^4 - 4y^2 + 4).$$

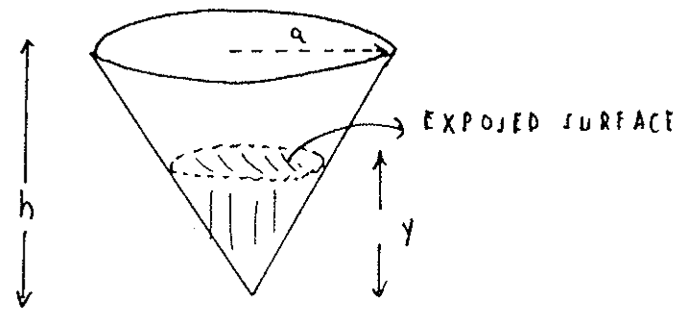
Calculate $\lim_{t\to\infty} y(t)$ for each of the three initial conditions: $y(0) = 2$, $y(0) = -1/2$ and $y(0) = -4$.

**PROBLEM 3:** (15 Points). Water flows into a conical water tank of radius $a$ and depth $h$ at a constant rate $k$ (with $k > 0$). Water is lost by evaporation at a rate proportional to the area of the exposed surface (see the picture below).

i) Derive an ODE for the depth $y(t)$ of water in the tank.

ii) Find the equilibrium level of water in the tank.

iii) The water will not overflow out of the tank when $k < k_c$, where $k_c$ is some critical value. Calculate $k_c$ explicitly.



# MATH 215 QUIZ # 1

**PROBLEM 1** $y' - \frac{p}{x} y = x$

integrating factor $\phi(x) = \exp\left[-\int^x p/s \, ds\right]$.

NOW $\int^x \frac{p}{s} ds = p \log x \longrightarrow \phi(x) = x^{-p}$.

HENCE MULTIPLY BY $x^{-p}$.

$$(x^{-p} y)' = x^{1-p}$$

$$x^{-p} y = \begin{cases} \frac{1}{2-p} x^{2-p} + C & p \neq 2 \\ \log x + C & p = 2 \end{cases}$$

THUS

$$\left.\begin{array}{ll} y = \frac{1}{2-p} x^2 + C x^p & \text{if } p \neq 2 \\ y = x^2(\log x + C) & \text{if } p = 2. \end{array}\right\} \text{general solution.}$$

NOTICE: $\lim_{x \to 0} x^2 \log x = 0$. THUS $y$ IS WELL-DEFINED AT $x = 0$ IF $p = 2$.

the general SOLUTION IS NOT WELL-DEFINED AT $x = 0$ IF $p < 0$, SINCE $x^p \to \infty$ AS $x \to 0$ IF $p < 0$.

**PROBLEM 2** $y' = \frac{x(y-1)^2}{x-1}$ $y(2) = 2$

equation IS separable: $\frac{x}{x-1} dx = (y-1)^{-2} dy$

NOW $\frac{x}{x-1} = \frac{(x-1)+1}{(x-1)} = 1 + \frac{1}{x-1}$.

THUS $\int^x \left(1 + \frac{1}{s-1}\right) ds = -(y-1)^{-1}$ upon integrating.

$\longrightarrow C + x + \log(x-1) = -(y-1)^{-1}$.

NOW $y(2) = 2 \longrightarrow C + 2 = -1 \rightarrow C = -3$

$$y - 1 = -\frac{1}{-3 + x + \log(x-1)} \longrightarrow y = 1 - \frac{1}{-3 + x + \log(x-1)}$$

NOW IF $y(2) = 1$ we have $C + 2 = -\frac{1}{0}$

WE would LIKE to set $C = \infty$. NOTICE THAT THE FUNCTION $y(x) \equiv 1$ SATISFIES THE INITIAL CONDITION AND THE ODE.

**PROBLEM 3** LET $v > 0$ IF BODY IS MOVING AWAY FROM EARTHS SURFACE AND $v < 0$ IF IT MOVES TOWARDS THE EARTH.

$v = 0$ WHEN $x = 3R$

THUS $m \frac{dv}{dt} = -\frac{mgR^2}{(x+R)}$

$x$: distance from surface of earth.

NOW $\frac{dv}{dt} = \frac{dv}{dx}\frac{dx}{dt} = v \frac{dv}{dx}$.

(take distance as measured from center)

$$v \frac{dv}{dx} = -\frac{mgR^2}{(x+R)^2} \longrightarrow \frac{1}{2} v^2 = \frac{mgR^2}{(x+R)} + C.$$

BUT $v = 0$ WHEN $x = 3R \longrightarrow \frac{mgR^2}{4R} = -C \rightarrow C = -\frac{mgR}{4}$.

HENCE $\frac{1}{2} v^2 = mgR^2\left[\frac{1}{x+R} - \frac{1}{4R}\right]$

need to take - square root since $v < 0$ IF IT goes down.

$\longrightarrow$ i) $v = -(2mgR^2)^{1/2}\left[\frac{1}{x+R} - \frac{1}{4R}\right]^{1/2}$

$x+R$: distance from center of earth.

NOW WHEN $x = 0$ at earth's surface we get

$$v = -(2gR^2)^{1/2}\left[\frac{1}{R} - \frac{1}{4R}\right]^{1/2}$$

ii) $v = -(2gR)^{1/2}(3/4)^{1/2} \longrightarrow v = -\frac{\sqrt{3}}{\sqrt{2}}(gR)^{1/2}$.

$\longrightarrow$ velocity at impact.

NOW return to (i)

$$v = -(2gR^2)^{1/2}\left[\frac{1}{x+R} - \frac{1}{4R}\right]^{1/2}$$

but $v = dx/dt$:

$$\frac{dx}{dt} = -(2gR^2)^{1/2}\left[\frac{1}{x+R} - \frac{1}{4R}\right]^{1/2}$$

NOTE: $x(0) = 3R$. FIND $T$ SUCH THAT $x(T) = 0$.

$$\left[\frac{1}{x+R} - \frac{1}{4R}\right]^{-1/2} dx = -(2gR^2)^{1/2} dt$$

$$\longrightarrow \int_{3R}^{x} \left[\frac{1}{s+R} - \frac{1}{4R}\right]^{-1/2} ds = -(2gR^2)^{1/2} \int_0^t ds$$

WANT $x = 0$ WHEN $t = T$. (time of impact).

$$\longrightarrow T = (2gR^2)^{-1/2} \int_0^{3R} \left[\frac{1}{s+R} - \frac{1}{4R}\right]^{-1/2} ds.$$

NOTE: we could have also interpreted the initial distance as being measured from the earth's center. IN which case $x(0) = 4R$.

# MATH 215 QUIZ # 2 SOLUTIONS

**PROBLEM 1** $x^2 y' = 4x^2 + 7xy + 2y^2$ WITH $y(1) = 0$

THIS GIVES $y' = 4 + 7y/x + 2y^2/x^2$

let $y = xv. \longrightarrow y' = xv' + v.$

$$xv' + v = 4 + 7v + 2v^2 \longrightarrow x v' = 2v^2 + 6v + 4 = 2(v^2 + 3v + 2)$$

$$\frac{dv}{(v+2)(v+1)} = \frac{2\,dx}{x} \qquad -\frac{1}{v+2} + \frac{1}{v+1} = \frac{1}{(v+1)(v+2)}$$

$$\left(\frac{1}{v+1} - \frac{1}{v+2}\right) dv = \frac{2\,dx}{x}$$

Integrate $\ln(|v+1|) - \ln(|v+2|) = 2\ln(xc)$

THUS $\ln\left|\frac{y+x}{y+2x}\right| = \ln(x^2c^2)$

$$\longrightarrow \left|\frac{y+x}{y+2x}\right| = x^2c^2.$$

WHEN $x = 1 \rightarrow y = 0 \qquad \frac{1}{2} = c^2 \longrightarrow c^2 = 1/2$

$$\left|\frac{y+x}{y+2x}\right| = \frac{x^2}{2}$$

NOW WE CAN UNDO $|\ \ |$ SIGN SINCE WHEN $x > 1$ $y > 0$.

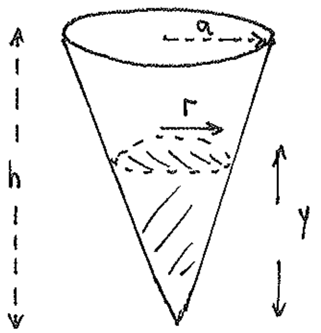
let $x > 1 \longrightarrow \frac{y+x}{y+2x} = \frac{x^2}{2}$

$$(y+x) = \frac{x^2}{2}(y+2x)$$

$$y\left[1 - x^2/2\right] = x^3 - x$$

$$y = \frac{x^3 - x}{1 - x^2/2} \qquad \text{DEFINED FOR } x < \sqrt{2}.$$

PROBLEM 3




$$dV/dt = K - \alpha A \qquad \alpha > 0$$

$a/h = r/y$ since we have similar triangles

$$V = \frac{1}{3}\pi r^2 y \qquad A = \pi r^2.$$

SINCE $r = \frac{a}{h} y$ THEN $V = \frac{1}{3}\pi \frac{a^2}{h^2} y^3$

$$dV/dt = \frac{\pi a^2}{h^2} y^2 \, dy/dt$$

$$A = \pi \frac{a^2}{h^2} y^2.$$

i) $$\frac{\pi a^2}{h^2} y^2 \, dy/dt = K - \alpha \frac{\pi a^2}{h^2} y^2$$

$$y(0) = y_0$$

ii) equilibrium is when $dy/dt = 0 \longrightarrow y_{EQ} = \left(\frac{K h^2}{\alpha \pi a^2}\right)^{1/2} > 0$

it is stable, as $\lim_{t \to \infty} y(t) = y_{EQ}$ FOR ANY INITIAL CONDITION.

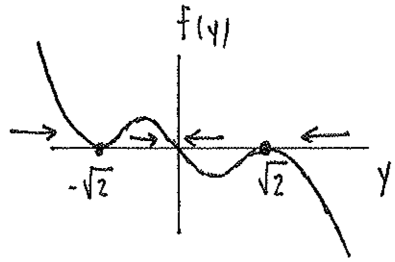
iii) FOR the water to not overflow we need $y_{EQ} < h$.

This means $\left(\frac{K h^2}{\alpha \pi a^2}\right)^{1/2} < h$

OR $K < \alpha \pi a^2$

$$K_c = \alpha \pi a^2$$

**PROBLEM 4** $\quad \dfrac{dy}{dt} = -y\,(y^4 - 4y^2 + 4)$




$$\frac{dy}{dt} = f(y) = -y\,(y^2 - 2)^2$$

EQ. POINTS AT $y = 0$, $y = \pm\sqrt{2}$.

FROM THE FIGURE WE HAVE

- IF $y(0) = 2$ then $y(t) \to \sqrt{2}$ AS $t \to \infty$
- IF $y(0) = -1/2$ then $y(t) \to 0$ AS $t \to \infty$
- IF $y(0) = -4$ then $y(t) \to -\sqrt{2}$ AS $t \to \infty$

MATH 215 QUIZ # 3

PROBLEM 1

$$y'' + 2y' + 2y = 0 \qquad y(\pi/4) = 2 \quad y'(\pi/4) = -2$$

let $y = e^{rx}$: $\quad r^2 + 2r + 2 = 0 \longrightarrow (r+1)^2 + 1 = 0 \qquad (r+1) = \pm i$

$r = -1 \pm i \qquad e^{-(1 \pm i)x}$ is a solution

$\longrightarrow \quad y_1 = e^{-x} \cos x \qquad y_2 = e^{-x} \sin x$

$y = c_1 y_1 + c_2 y_2$. gen. solution

$$y = e^{-x} ( c_1 \cos x + c_2 \sin x )$$

much more convenient to write

$$y = e^{-(x-\pi/4)} \left( b_1 \cos ( x - \pi/4 ) + b_2 \sin ( x - \pi/4 ) \right).$$

$y(\pi/4) = 2 \longrightarrow b_1 = 2$

$y'(\pi/4) = -b_1 + b_2 = -2 \longrightarrow b_2 = 0$

$$y = 2e^{-(x-\pi/4)} \cos (x - \pi/4).$$

PROBLEM 2 $\qquad$ let $y = x^p$, $\quad y' = px^{p-1}$, $\quad y'' = p(p-1)x^{p-2}$.

$x^2 y'' + 3xy' + y = 0$

$\longrightarrow p(p-1) + 3p + 1 = 0$

$p^2 + 2p + 1 = 0 \qquad (p+1)^2 = 0 \qquad p = -1.$ $\qquad$ (5 pts for this)

so $\quad y_1 = 1/x$ is a solution.

now let $y = x^{-1} v$.

$$y' = -x^{-2} v + x^{-1} v' \qquad y'' = 2x^{-3} v - 2x^{-2} v' + x^{-1} v''$$

substitute into the ODE to get

$$x^2 ( x^{-1} v'' - 2x^{-2} v' + 2x^{-3} v ) + 3x ( x^{-1} v' - x^{-2} v ) + x^{-1} v = 0$$

$$\longrightarrow x v'' - 2v' + 3v' = 0$$

$$\longrightarrow x v'' + v' = 0 \longrightarrow (xv')' = 0 \longrightarrow xv' = c_1$$

THUS $\quad v' = C_1/x$

$\rightarrow \quad v = C_1 \log x + C_2 .$

THUS $\quad y = \frac{1}{x} v \quad \longrightarrow \quad y = \frac{C_1}{x} \log x + C_2/x$ IS GENERAL SOLUTION.

PROBLEM 3 $\quad y'' + p y' + y = 1 + e^{-x} \qquad y(0) = y'(0) = 0$

Homog. problem $\quad y'' + p y' + y = 0$

let $y = e^{rx} \longrightarrow r^2 + pr + 1 = 0$

$$r_{\pm} = \frac{-p \pm (p^2 - 4)^{1/2}}{2}$$

3

particular solution $\quad y_{p_1}'' + p y_{p_1}' + y_{p_1} = 1 \longrightarrow y_{p_1} = 1$

2

NOW $\quad y_{p_2}'' + p y_{p_2}' + y_{p_2} = e^{-x}$

try $\quad y_{p_2} = A e^{-x}$

$\longrightarrow \quad A - pA + A = 1$

$\longrightarrow \quad A = \frac{1}{2-p} \qquad p \neq 2.$

$\lim\limits_{x \to \infty} y(x) = 1$ FOR $p > 0$

3

IF $p = 2$ NOT valid.

$$y_{p_2} = \frac{1}{2-p} e^{-x} \qquad p \neq 2.$$

2

NOTICE IF $p = 2$ then $r_{\pm} = -1$ which is a repeated root OF multiplicity 2. HENCE $y_{p_2}$ must have a different FORM.

CASE 1 $\quad 0 \le p < 2. \qquad r_{\pm} = \frac{-p \pm i(4-p^2)^{1/2}}{2}$

$$\longrightarrow \quad y_1 = e^{-px/2} \cos\left[\frac{(4-p^2)^{1/2}}{2} x\right] \qquad y_2 = e^{-px/2} \sin\left[\frac{(4-p^2)^{1/2}}{2} x\right]$$

THUS $$y = e^{-px/2}\left[C_1 \cos\left(\frac{(4-p^2)^{1/2}}{2} x\right) + C_2 \sin\left(\frac{(4-p^2)^{1/2}}{2} x\right)\right] + 1 + \frac{e^{-x}}{2-p}$$

(gen. solution)

NOW $Y(0)=0$

$$\to \quad C_1 + 1 + \frac{1}{2-p} = 0 \quad \longrightarrow \quad C_1 = \frac{1}{p-2} - 1 = \frac{3-p}{p-2}$$

$Y'(0) = 0$

$$\boxed{C_1 = -1 + 1/_{p-2}}$$

$$\to \quad -\frac{p}{2} C_1 + \frac{(4-p^2)^{1/2}}{2} C_2 + \frac{1}{p-2} = 0$$

$$\frac{(4-p^2)^{1/2}}{2} C_2 = \frac{p}{2}\left(\frac{3-p}{p-2}\right) - \frac{1}{p-2} = \frac{1}{p-2}\left[\frac{1}{2}p(3-p) - 1\right]$$

$$\longrightarrow C_2 = \frac{1}{(p-2)(4-p^2)^{1/2}}\left[p(3-p) - 2\right] = \frac{(1-p)}{(4-p^2)^{1/2}}$$

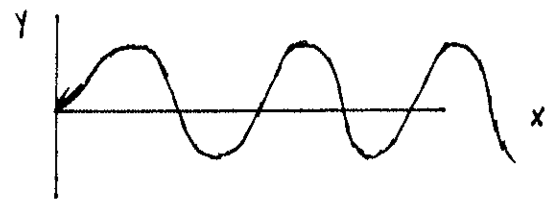
$$\boxed{C_2 = \frac{(1-p)}{(4-p^2)^{1/2}}}$$

IF $p=0$, then

$$Y = C_1 \cos X + C_2 \sin X + 1 + e^{-X}/_2$$

$C_1 = -3/_2 \qquad C_2 = 1/_2$ (substitute $p=0$ into boxed formulas

NOW we get a graph sort of like




5

NOTICE $\lim_{X \to \infty} Y(X) = 1$ FOR ANY $p$ WITH $2 > p > 0$.

(solution to homog. problem tend to zero as $X \to \infty$ for any $p > 0$). Also $Y_{p_1} = 1$ AND $Y_{p_2} = \frac{1}{2-p} e^{-X}$, IF $p \neq 2$.

· There is no limiting behavior of $Y(X)$ as $X \to \infty$ when $p=0$

CASE 2 $p > 2$.

$$r_\pm = \frac{-p \pm (p^2-4)^{1/2}}{2} \to \text{real}$$

$$Y = C_1 e^{r_+ X} + C_2 e^{r_- X} + 1 + \frac{e^{-X}}{2-p}$$

$$Y(0) = 0 \quad \longrightarrow \quad C_1 + C_2 + 1 + \frac{1}{2-p} = 0$$

$$Y'(0) = 0 \quad \longrightarrow \quad r_+ C_1 + r_- C_2 - \frac{1}{2-p} = 0$$

$$\left.\begin{array}{l} r_+ C_1 + r_+ C_2 + r_+ + r_+/_{2-p} = 0 \\ r_+ C_1 + r_- C_2 - 1/_{2-p} = 0 \end{array}\right\} \text{ subtract } \rightarrow (r_+ - r_-) C_2 = - r_+ - r_+/_{2-p} + 1/_{p-2}$$

$$C_2 = \frac{-1}{r_+ - r_-}\left[ r_+ + r_+/_{2-p} + 1/_{2-p}\right] \quad 5$$

NOW
$$\left.\begin{array}{l} r_- C_1 + r_- C_2 + r_- + r_-/_{2-p} = 0 \\ r_+ C_1 + r_- C_2 + -1/_{2-p} = 0 \end{array}\right\} \text{ subtract } \longrightarrow (r_- - r_+) C_1 = - r_- - r_-/_{2-p} - 1/_{2-p}$$

clearly since $r_+ > 0$ and $r_- > 0$ then $y \to 1$ as $x \to \infty$.

<u>CASE 3</u> $p = 2$.

IF $p = 2$ then $r_+ = r_- = -1$

so $Y_1 = e^{-x}$ is a solution.

the second solution is $Y_2 = x e^{-x}$. 5

NOW TO FIND $Y_{P_2}$: $Y_{P_2}'' + 2 Y_{P_2}' + Y_{P_2} = e^{-x}$

let $Y_{P_2} = v e^{-x}$.

$Y_{P_2}' = v' e^{-x} - v e^{-x}$ $\quad Y_{P_2}'' = v'' e^{-x} - 2 v' e^{-x} + v e^{-x}$

substitute to get $\longrightarrow v'' - \cancel{2v'} + \cancel{v} + \cancel{2v'} - \cancel{2v} + \cancel{v} = 1$

$\longrightarrow v'' = 1$ choose $v = x^2/_2$.

$\longrightarrow Y_{P_2} = \frac{x^2}{2} e^{-x}$.

HENCE $Y = C_1 e^{-x} + C_2 x e^{-x} + \frac{x^2}{2} e^{-x} + 1$

$Y(0) = 0 \longrightarrow C_1 + 1 = 0 \quad C_1 = -1$

$Y'(0) = 0 \longrightarrow -C_1 + C_2 = 0 \quad C_2 = C_1 = -1$

$Y = -e^{-x} - x e^{-x} + \frac{x^2}{2} e^{-x} + 1$

clearly $\lim_{x \to \infty} Y(x) = 1$