MATH 215/255, SECTION 102, MIDTERM II,
NOVEMBER 19 2012 (H. DIXIT)
Closed Book and Notes. Calculators not allowed.
50 Minutes. Total: 40 marks
This exam contains two pages along with a table of Laplace transforms.

<u>**Problem 1:**</u> [12 marks] Short answers. Detailed explanation not required. 3 marks for each question.

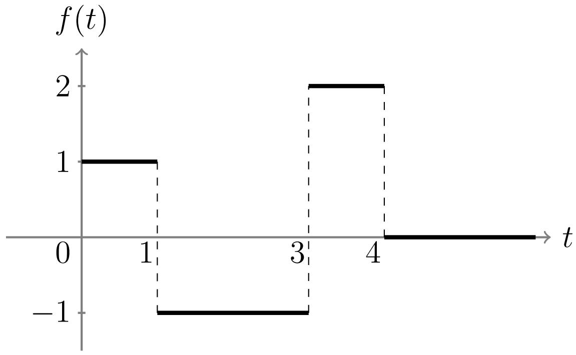
i) Find the particular solution of the non-homogeneous equation

$$y'' + y' + 3y = t.$$

ii) Find the damping constant, γ, such that the following spring-mass-damper system is critically damped:

$$2y'' + \gamma y' + 8y = 0.$$

iii) Express the function, $f(t)$, shown in the figure below in terms of unit step functions.



iv) Find the inverse Laplace transform of

$$F(s) = \frac{2}{s^2 + 3s - 4}$$

<u>**Problem 2:**</u> [10 points] If $y_1(t) = (1 + t)$ and $y_2(t) = e^t$ are solutions of the homogeneous part of the following differential equation,

$$ty'' - (1 + t)y' + y = t^2 e^{2t}, \qquad t > 0,$$

find the particular solution using variation of parameters.

Problem 3: [10 points] Consider a spring-mass-damper system forced by an external forcing with frequency ω. The position of the mass is described by the following initial value problem:

$$u'' + 5u' + 6u = 3\cos(\omega t), \quad u(0) = 1, u'(0) = 0.$$

i) Determine the steady state or forced response of the system. [7 points]

ii) The steady state response can be expressed as $R\cos(\omega t - \delta)$. Determine the amplitude R in terms of ω. [3 points]

Problem 4: [8 points] Consider the following differential equation with discontinuous forcing:

$$y'' + 3y' + 2y = \begin{cases} 1 & 0 \le t < 10, \\ 0 & t \ge 10, \end{cases} \quad ; \quad y(0) = y'(0) = 0,$$

i) Calculate the solution $y(t)$ using Laplace transforms. [6 points]

ii) Determine $\lim_{t\to\infty} y(t)$. [2 points]

Table of Elementary Laplace Transforms

	$f(t) = \mathcal{L}^{-1}\{F(s)\}$	$F(s) = \mathcal{L}\{f(t)\}$
1.	1	$\dfrac{1}{s}, \quad s > 0$
2.	e^{at}	$\dfrac{1}{s-a}, \quad s > a$
3.	$t^n, \quad n =$positive integer	$\dfrac{n!}{s^{n+1}}, \quad s > 0$
4.	$t^p, \quad p > -1$	$\dfrac{\Gamma(p+1)}{s^{p+1}}, \quad s > 0$
5.	$\sin(at)$	$\dfrac{a}{s^2+a^2}, \quad s > 0$
6.	$\cos(at)$	$\dfrac{s}{s^2+a^2}, \quad s > 0$
7.	$\sinh(at)$	$\dfrac{a}{s^2-a^2}, \quad s > \lvert a\rvert$
8.	$\cosh(at)$	$\dfrac{s}{s^2-a^2}, \quad s > \lvert a\rvert$
9.	$e^{at}\sin(bt)$	$\dfrac{b}{(s-a)^2+b^2}, \quad s > a$
10.	$e^{at}\cos(bt)$	$\dfrac{s-a}{(s-a)^2+b^2}, \quad s > a$
11.	$t^n e^{at}, \quad n =$positive integer	$\dfrac{n!}{(s-a)^{n+1}}, \quad s > a$
12.	$u_c(t)$	$\dfrac{e^{-cs}}{s}, \quad s > 0$
13.	$u_c(t)f(t-c)$	$e^{-cs}F(s)$
14.	$e^{ct}f(t)$	$F(s-c)$
15.	$f(ct)$	$\dfrac{1}{c}F\left(\dfrac{s}{c}\right), \quad c > 0$
16.	$\displaystyle\int_0^t f(t-\tau)g(\tau)d\tau$	$F(s)G(s)$
17.	$\delta(t-c)$	e^{-cs}
18.	$f^{(n)}(t)$	$s^nF(s) - s^{n-1}f(0) - \cdots - f^{(n-1)}(0)$
19.	$(-1)^n f(t)$	$F^{(n)}(s)$

MATH 215/255 MIDTERM II SOLUTIONS (H. Dixit)

Problem 1:

(i) $y'' + y' + 3y = t$

Find particular solution.

Using method of undetermined coefficients, let

$$y_p = At + B$$

$$\Rightarrow \quad y_p' = A$$

$$y_p'' = 0$$

$$\Rightarrow \quad 0 + A + 3(At+B) = t \quad \Rightarrow \quad \left.\begin{array}{l} 3A = 1 \\ \& \ A + 3B = 0 \end{array}\right\} \begin{array}{l} A = \frac{1}{3} \\ B = \frac{-1}{9} \end{array}$$

$$\therefore \quad \boxed{y_p = \frac{t}{3} - \frac{1}{9}}$$

(ii) $2y'' + \gamma y' + 8y = 0$

Find γ such that the system is critically damped.

characteristic equation: $2\lambda^2 + \gamma\lambda + 8 = 0$

$$\Rightarrow \quad \lambda = \frac{-\gamma \pm \sqrt{\gamma^2 - 4\cdot 2\cdot 8}}{4}$$

$$= \frac{-\gamma \pm \sqrt{\gamma^2 - 64}}{4}$$

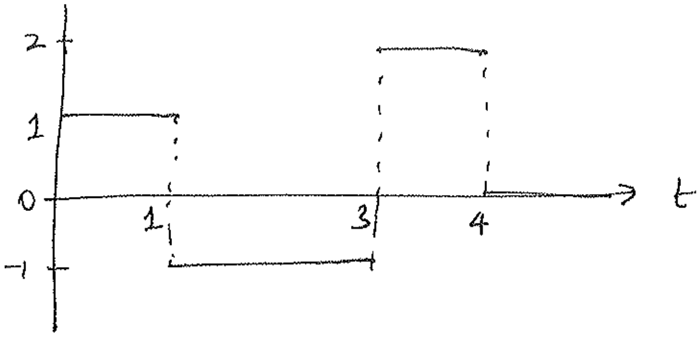
System critically damped when $\gamma^2 - 64 = 0 \quad \Rightarrow \quad \gamma = 8$

Note: If $\gamma < 8$, λ's are complex conjugates : Underdamped

If $\gamma > 8$, λ's are real & negative : Over damped.

(iii)

$$f(t) = \begin{cases} 1 & 0 \le t < 1 \\ -1 & 1 \le t < 3 \\ 2 & 3 \le t < 4 \\ 0 & t \geq 4 \end{cases}$$



$$\Rightarrow f(t) = 1 + \begin{cases} 0 & 0 \le t < 1 \\ -2 & 1 \le t < 3 \\ 1 & 3 \le t < 4 \\ -1 & t \geq 4 \end{cases}$$

Recall

$$u_c(t) = \begin{cases} 0 & t \leq c \\ 1 & t > c \end{cases}$$

$$= 1 + (-2)u_1(t) + 3\,u_3(t) - 2\,u_4(t)$$

$$= 1 - 2u_1(t) + 3u_3(t) - 2u_4(t)$$

(iv)

$$F(s) = \frac{2}{s^2 + 3s - 4}$$

$$= \frac{2}{(s+4)(s-1)} = 2\left[\frac{A}{s+4} + \frac{B}{s-1}\right]$$

$$\Rightarrow A(s-1) + B(s+4) = 1 \Rightarrow \left.\begin{matrix} A + B = 0 \\ -A + 4B = 1 \end{matrix}\right\} \begin{matrix} A = \frac{-1}{5} \\ B = \frac{1}{5} \end{matrix}$$

$$\therefore F(s) = \frac{2}{5}\left[\frac{-1}{s+4} + \frac{1}{s-1}\right]$$

$$\Rightarrow f(t) = \frac{2}{5}\left[-e^{-4t} + e^{t}\right]$$

Problem 2:

$$ty'' - (1+t)y' + y = t^2 e^{2t}, \quad t>0$$

$$\left.\begin{array}{l} y_1(t) = 1+t \\ y_2(t) = e^t \end{array}\right\} \text{Homogeneous solutions}$$

$$\Rightarrow \quad y_{homo.} = c_1(1+t) + c_2 e^t$$

Particular solution:- Let $y_p = u_1(t)(1+t) + u_2(t)e^t$

$$y_p' = u_1'(1+t) + u_1 + u_2' e^t + u_2 e^t$$

Set $\boxed{u_1'(1+t) + u_2' e^t = 0}$ — 1

$$\Rightarrow \quad y_p' = u_1 + u_2 e^t$$

$$y_p'' = u_1' + u_2' e^t + u_2 e^t$$

Substituting, we have

$$t\left[u_1' + u_2' e^t + u_2 e^t\right] - (1+t)\left[u_1 + u_2 e^t\right] + \left[u_1(1+t) + u_2 e^t\right] = t^2 e^{2t}$$

$$\Rightarrow \quad u_1\left[-\cancel{(1+t)} + \cancel{(1+t)}\right] + u_2\left[-\cancel{(1+t)e^t} + \cancel{te^t} + \cancel{e^t}\right] + tu_1' + u_2' te^t = t^2 e^{2t}$$

$$\Rightarrow \quad \boxed{u_1' + u_2' e^t = te^{2t}} \quad — 2$$

Solving for u_1' & u_2' from 1 & 2 :-

$$\begin{bmatrix} 1+t & e^t \\ 1 & e^t \end{bmatrix} \begin{bmatrix} u_1' \\ u_2' \end{bmatrix} = \begin{bmatrix} 0 \\ te^{2t} \end{bmatrix}$$

$$\begin{bmatrix} u_1' \\ u_2' \end{bmatrix} = \frac{\begin{bmatrix} e^t & -e^t \\ -1 & 1+t \end{bmatrix}}{(1+t)e^t - e^t} \begin{bmatrix} 0 \\ te^{2t} \end{bmatrix}$$

$$= \frac{1}{te^t} \begin{bmatrix} e^t & -e^t \\ -1 & 1+t \end{bmatrix} \begin{bmatrix} 0 \\ te^{2t} \end{bmatrix}$$

$$\Rightarrow \quad u_1' = \frac{1}{te^t} \times -te^{3t} = \cancel{e^{3t}} - e^{2t}$$

$$\& \quad u_2' = \frac{1}{te^t} \times (1+t)te^{2t} = (1+t)e^t$$

$$\therefore \quad \cancel{u_1 = -t + c_1} \qquad u_1 = -\frac{e^{2t}}{2} + c_1$$

$$\cancel{\text{and } u_2 =} \qquad u_2 = te^t + c_2$$

$$\therefore \quad y_p = \left(-\frac{e^{2t}}{2} + c_1\right)(1+t) + (te^t + c_2)e^t$$

$$= \underbrace{c_1(1+t) + c_2 e^t}_{\text{same as homogeneous solution}} - \frac{(1+t)}{2}e^{2t} + te^{2t}$$

$\Rightarrow$ Particular solution : $y_p = \left(\frac{t-1}{2}\right)e^{2t}$

Problem 3:

$$u'' + 5u' + 6u = 3\cos(\omega t) \; ; \qquad u(0) = 1$$
$$u'(0) = 0.$$

(i) Steady state response $\Longrightarrow$ Particular solution.

Using method of undetermined coefficients, let

$$u_p = A\cos(\omega t) + B\sin(\omega t)$$
$$u_p' = -A\omega\sin(\omega t) + B\omega\cos(\omega t)$$
$$u_p'' = -A\omega^2\cos(\omega t) - B\omega^2\sin(\omega t)$$

Substituting, we have

$$\left[-A\omega^2\cos(\omega t) - B\omega^2\sin(\omega t)\right] + 5\left[-A\omega\sin(\omega t) + B\omega\cos(\omega t)\right]$$
$$+ 6\left[A\cos(\omega t) + B\sin(\omega t)\right] = 3\cos(\omega t)$$

$O(\cos(\omega t))$: $\quad -A\omega^2 + 5B\omega + 6A = 3 \quad \rightarrow$

$O(\sin\omega t)$: $\quad -B\omega^2 - 5A\omega + 6B = 0 \longrightarrow (6-\omega^2)B = 5A\omega$

$$\Longrightarrow A = \frac{6-\omega^2}{5\omega}B$$

$$\Longrightarrow (6-\omega^2)\frac{6-\omega^2}{5\omega}B + 5\omega B = 3$$

$$\Longrightarrow \left[(6-\omega^2)^2 + 25\omega^2\right]B = 15\omega$$

$$\Longrightarrow B = \frac{15\omega}{\left[(6-\omega^2)^2 + 25\omega^2\right]}$$

$$\therefore A = \frac{3(6-\omega^2)}{\left[(6-\omega^2)^2 + 25\omega^2\right]}$$

$\therefore$ Steady state response:

$$u_p = \frac{3(6-\omega^2)\cos(\omega t) + 15\omega\sin(\omega t)}{(6-\omega^2)^2 + 25\omega^2}$$

(ii) Let $u_p = R\cos(\omega t - \delta)$

$= R\cos\delta \cos(\omega t) + R\sin\delta \cdot \sin(\omega t)$

$$\Rightarrow \quad R^2 = A^2 + B^2$$

$$= \frac{9(6-\omega^2)^2}{[(6-\omega^2)^2 + 25\omega^2]^2} + \frac{225\omega^2}{[(6-\omega^2)^2 + 25\omega^2]}$$

$$= \frac{9[(6-\omega^2)^2 + 25\omega^2]}{[(6-\omega^2)^2 + 25\omega^2]^2}$$

$$\therefore \quad R = \frac{3}{\sqrt{(6-\omega^2)^2 + 25\omega^2}}$$

Note that if the damping was zero, the differential equation becomes

$$u'' + 6u = 3\cos(\omega t).$$

Solving for amplitude R, we get

$$R = \frac{3}{(6-\omega^2)}$$

Resonance

$\therefore$ As $\omega \to \sqrt{6}$, $R \to \infty$:

Problem ④:

$$y'' + 3y' + 2y = \begin{cases} 1 & 0 \le t < 10 \\ 0 & t \geq 10 \end{cases} \qquad y(0) = y'(0) = 0$$

(i) Taking Laplace transform, we have

$$\left[s^2 Y(s) - s y(0) - y'(0)\right] + 3\left[s Y(s) - y(0)\right] + 2 Y(s)$$

$$= \mathcal{L}\{g(t)\}$$

where $g(t) = \begin{cases} 1 & 0 \le t < 10 \\ 0 & t \ge 10 \end{cases}$

$$= 1 + \begin{cases} 0 & 0 \le t < 10 \\ -1 & t \ge 10 \end{cases}$$

$$= 1 - u_{10}(t)$$

$$\therefore \mathcal{L}\{g(t)\} = \frac{1}{s} - \frac{e^{-10s}}{s}$$

$$\Rightarrow (s^2 + 3s + 2) Y(s) = \frac{1}{s} - \frac{e^{-10s}}{s}$$

$$\Rightarrow Y(s) = H(s) - e^{-10s} H(s)$$

where $H(s) = \dfrac{1}{s(s^2+3s+2)}$

$$= \frac{1}{s(s+1)(s+2)}$$

$$= \frac{1}{2s} - \frac{1}{s+1} + \frac{1}{2(s+2)}$$

$$h(t) = \mathcal{L}^{-1}\{H(s)\} = \frac{1}{2} - e^{-t} + \frac{1}{2} e^{-2t}$$

$$y(t) = \mathcal{L}^{-1}\{Y(s)\} = h(t) - u_{10}(t)\, h(t-10)$$

$$\Rightarrow y(t) = \left(\frac{1}{2} - e^{-t} - \frac{1}{2} e^{-2t}\right) - u_{10}(t)\left[\frac{1}{2} - e^{-(t-10)} + \frac{1}{2} e^{-2(t-10)}\right]$$

(ii) As $t \to \infty$; $e^{-t} \to 0$

$e^{-2t} \to 0$

& $u_{10}(t) = 1$

$$\Rightarrow \lim_{t\to\infty} y(t) = \frac{1}{2} - 1\times\left[\frac{1}{2}\right] = 0$$