MATH 215/255, SECTION 102, MIDTERM I,
OCTOBER 12 2012 (H. DIXIT)
Closed Book and Notes. 50 Minutes. Total: 40 points
This exam contains two pages.

Problem 1: [12 points] State whether the following statements are **True** or **False**. **Justify your answer with a clear explanation.**

i) The following equation is an exact equation [3 points].

$$\frac{dy}{dx} = \frac{2x + 3y}{3x + 2y}$$

ii) The two functions $y_1(t) = e^{-2t}$ and $y_2(t) = e^t$ form a fundamental set of solutions of the differential equation $y'' + y' - 2y = 0$. [3 points]

iii) For the following differential equation, $y = 1$ is a stable equilibrium.

$$\frac{dy}{dx} = (y - 1)(y - 2).$$

(Note: Can verify true or false without solving the equation). [3 points]

iv) If $f(t)$, $g(t)$ and $h(t)$ are three differentiable functions, then $W[fg, fh](t) = f^2 W[g, h](t)$. Here W is the Wronskian. [3 points]

Problem 2: [7 points] Consider the following differential equation:

$$(t - 3)y' + 2y = \frac{1}{t - 3}, \qquad y(0) = 1.$$

i) Without solving the differential equation, obtain the interval of existence. [2 points]

ii) Now obtain a solution of the differential equation. Does the interval of existence agree with that obtained in part (i)? [5 points]

Problem 3: [9 points] Consider the following differential equation:

$$y'' - 2y' - 3y = 0.$$

i) Obtain the general solution. [4 points]

ii) If $y(0) = \alpha$ and $y'(0) = \beta$, obtain the exact solution in terms of α and β. [3 points]

iii) What should be the relation between α and β such that $y(t) \to 0$ as $t \to \infty$. [2 points]

Problem 4: [12 points] Consider the following differential equation for fish population, $y(t)$:

$$\frac{dy}{dt} = f(y), \quad \text{where } f(y) = \sqrt{y} - \frac{y}{2}; \quad y > 0, t > 0.$$

i) Determine the equilibrium solutions. [2 points]

ii) Plot $f(y)$ versus y and use this to plot the solution curves, i.e. y versus t for several initial conditions. Comment on the stability of the equilibrium solutions. [5 points]

iii) Now solve the differential equation with the initial condition $y(0) = 1$. Verify that $\lim_{t\to\infty} = y_e$ where y_e is an equilibrium solution. [5 points]

MIDTERM - I SOLUTIONS

12 October 2012

(1) TRUE OR FALSE QUESTIONS

(i) $$\frac{dy}{dx} = \frac{2x+3y}{3x+2y}$$ $\boxed{\text{Ans: FALSE}}$

$$\Rightarrow \quad (2x+3y) - (3x+2y)\frac{dy}{dx} = 0$$

$$M + N\frac{dy}{dx} = 0$$

$$M = 2x+3y \quad \Rightarrow \quad \frac{\partial M}{\partial y} = 3$$

$$N = -(3x+2y) \quad \Rightarrow \quad \frac{\partial N}{\partial x} = -3$$

Since $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$, the equation is NOT exact.

(ii) $y_1 = e^{-2t}$ $\boxed{\text{Ans: TRUE}}$

$y_2 = e^{t}$

Equation $y'' + y' - 2y = 0$

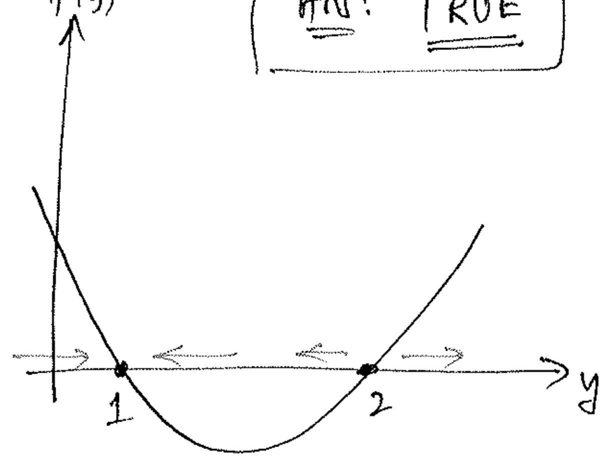
y_1 & y_2 are solutions of the above equation.

$$W[y_1, y_2](t) = \begin{vmatrix} y_1(t) & y_2(t) \\ y_1'(t) & y_2'(t) \end{vmatrix} = \begin{vmatrix} e^{-2t} & e^{t} \\ -2e^{-2t} & e^{t} \end{vmatrix}$$

$$= e^{-t} + 2e^{-t} = 3e^{-t} \neq 0$$

Since $W \neq 0$, y_1 & y_2 form a fundamental set of solutions.

(iii) $\dfrac{dy}{dt} = \underbrace{(y-1)(y-2)}_{f(y)}$



Ans: TRUE

$y=1$ is a stable equilibrium.

(iv) $W[fg, fh](t) = \begin{vmatrix} fg & fh \\ fg'+f'g & fh'+f'h \end{vmatrix}$ where $f' = \dfrac{df}{dt}$, $g' = \dfrac{dg}{dt}$, $h' = \dfrac{dh}{dt}$

$= (fh'+f'h)\,fg - fh\,(fg'+f'g)$

$= f^2(h'g - hg')$

$= f^2 \begin{vmatrix} g & h \\ g' & h' \end{vmatrix}$

Ans: TRUE

$\Rightarrow W[fg, fh](t) = f^2\, W[g,h](t)$

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(2) $(t-3)y' + 2y = \dfrac{1}{t-3}$; $y(0) = 1$

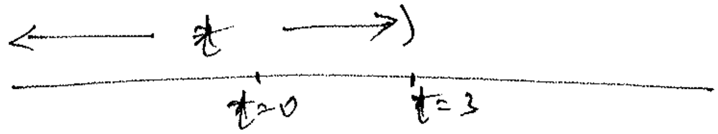
(i) Writing the equation in standard form:

$y' + p(t)y = q(t)$, we have

$$y' + \left(\frac{2}{t-3}\right) y = \frac{1}{(t-3)^2}$$

$p(t)$ and $q(t)$ are discontinuous at $t=3$.

Since the initial condition is prescribed at $t=0$, the interval of existence is $-\infty < t < 3$



(ii) Integrating factor is $\phi = e^{\int \frac{2}{t-3}\,dt}$

$$= e^{2\ln|t-3|} = (t-3)^2$$

$$\Rightarrow y'(t-3)^2 + 2(t-3)\,y = 1$$

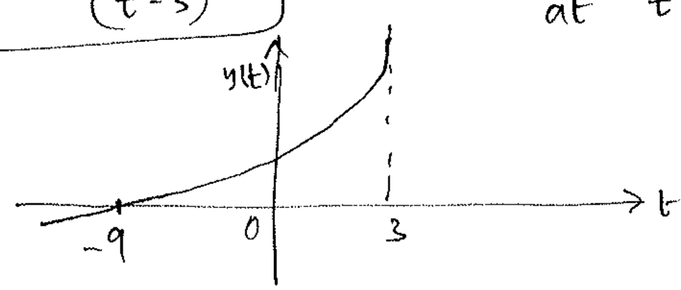
$$\Rightarrow \frac{d}{dt}\left[y(t-3)^2\right] = 1$$

$$\Rightarrow y(t-3)^2 = t+c \Rightarrow \boxed{y = \frac{t+c}{(t-3)^2}}$$

Now $y(0)=1 \Rightarrow 1 = \frac{c}{9} \Rightarrow c=9$

$$\therefore \boxed{y(t) = \frac{t+9}{(t-3)^2}} \leftarrow$$

Solution discontinuous at $t=3$



(3) $$y'' - 2y' - 3y = 0$$

(i) General Solution:-

Let $y = e^{rt}$

$\Rightarrow y' = re^{rt}$

$y'' = r^2 e^{rt}$

$\Rightarrow (r^2 - 2r - 3) e^{rt} = 0$

Characteristic equation: $r^2 - 2r - 3 = 0$

$\Rightarrow (r-3)(r+1) = 0$

$\Rightarrow r_1 = 3, \quad r_2 = -1$

$\therefore y_1(t) = e^{3t} \quad ; \quad y_2 = e^{-t}$

General Solution:- $y(t) = C_1 e^{3t} + C_2 e^{-t}$

(ii) Now $y(0) = \alpha$, $y'(0) = \beta$ $\Rightarrow$ $\alpha = C_1 + C_2$, $\beta = 3C_1 - C_2$

$\therefore 4C_1 = \alpha + \beta \Rightarrow C_1 = \frac{\alpha+\beta}{4}$,

$C_2 = \alpha - \left(\frac{\alpha+\beta}{4}\right) = \frac{3\alpha - \beta}{4}$

$$\therefore y(t) = \left(\frac{\alpha+\beta}{4}\right) e^{3t} + \left(\frac{3\alpha-\beta}{4}\right) e^{-t}$$

$$\therefore \quad y(t) = \left(\frac{\alpha+\beta}{4}\right)e^{3t} + \left(\frac{3\alpha-\beta}{4}\right)e^{-t}$$

(iii) As $t \to \infty$, $e^{-t} \to 0$, but e^{3t} grows unboundedly.

Therefore, for $y(t) \to 0$ as $t \to \infty$, we require

$$\frac{\alpha+\beta}{4} = 0 \quad \Rightarrow \quad \boxed{\alpha+\beta = 0}$$

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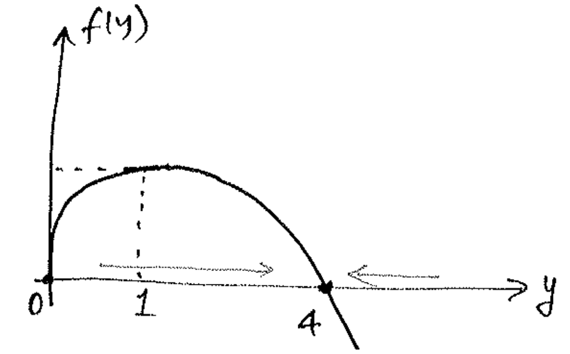
(4) $$\frac{dy}{dt} = f(y) \qquad \text{where} \quad f(y) = \sqrt{y} - \frac{y}{2}$$

$$y \geq 0, \quad t > 0.$$

(i) Equilibrium Solutions :-

$$f(y) = 0 \quad \Rightarrow \quad \sqrt{y} - \frac{y}{2} = 0$$

$$\Rightarrow \quad y = 0 \quad \text{and} \quad y = 4$$



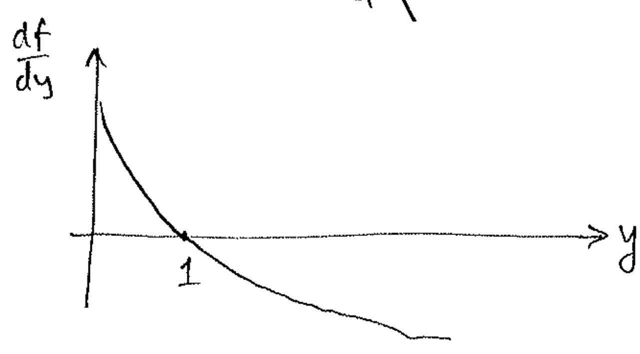
(ii) $$\frac{df}{dy} = \frac{1}{2\sqrt{y}} - \frac{1}{2}$$

$$\frac{df}{dy} = 0 \quad \Rightarrow \quad y = 1$$

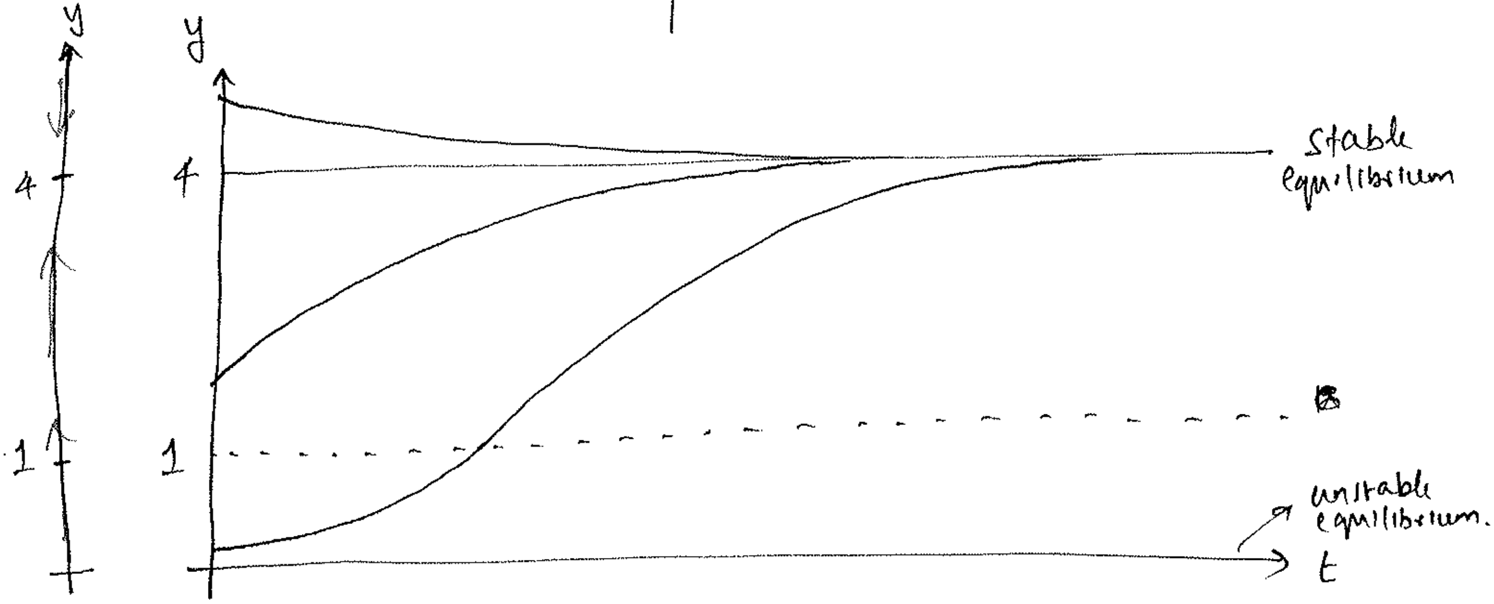
At $y = 0$; $\dfrac{df}{dy} \to \infty$

At $y = 1$; $\dfrac{df}{dy} = 0$

Plot of $\dfrac{df}{dy}$ vs y :



Interval	$\frac{dy}{dt} = f(y)$	$\frac{df}{dy}$	$\frac{d^2y}{dt^2} = \frac{df}{dy} \cdot f(y)$	
$(0,1)$	>0	>0	>0	: Concave up
$(1,4)$	>0	<0	<0	: Concave down
$(4,\infty)$	<0	<0	>0	: concave up.



(iii) Solution: $\frac{dy}{dt} = \sqrt{y} - \frac{y}{2}$

$y(0) = 1$

$$\Rightarrow \frac{dy}{\sqrt{y} - \frac{y}{2}} = dt$$

$$\Rightarrow \frac{2\,dy}{\sqrt{y}(2-\sqrt{y})} = dt$$

Let $\sqrt{y} = u \Rightarrow y = u^2$

$\Rightarrow dy = 2u\,du$

$$\Rightarrow \quad \frac{2 \cdot 2\cancel{u}\, du}{\cancel{u}\,(2-u)} = dt$$

$$\Rightarrow \quad 2 \cdot 2 \int \frac{du}{2-u} = \int dt$$

$$\Rightarrow \quad \frac{2 \cdot 2 \ln|2-u|}{(-1)} = t + c_1$$

$$\Rightarrow \quad \ln|2-u| = \frac{-t-c_1}{4}$$

or $\quad 2-u = e^{\frac{-t-c_1}{4}}$

$$= e^{\frac{-t}{4}} \cdot e^{\frac{-c_1}{4}}$$

Let $\quad C = e^{\frac{-c_1}{4}}$: arbitrary constant.

$$\Rightarrow \quad 2-u = Ce^{-t/4} \quad \text{or} \quad \boxed{u = 2 - Ce^{-t/4}}$$

Now $\quad y(0) = 1$

$\Rightarrow \quad u(0) = 1 \quad \Rightarrow \quad 1 = 2 - C \quad \Rightarrow \quad C = 1$

$$\therefore \quad u(t) = 2 - e^{-t/4}$$

$$\therefore \quad \sqrt{y} = 2 - e^{-t/4}$$

or $\quad \boxed{y = \left(2 - e^{-t/4}\right)^2}$

Clearly, at $t = 0$, $y = 1$

and as $t \to \infty$; $\quad y = \lim_{t\to\infty}\left(2 - e^{-t/4}\right)^2 = 4$

$$= y_e$$