MATH 215, SECTION 202, MIDTERM I: FEBRUARY 13 2012 (H. DIXIT)

Closed Book and Notes. 50 Minutes. Total 40 points

Problem 1: [20 points] Solve the following problems for $y(x)$:

i) $y' + y = a\sin(x)$, with $y(0) = 0$. (Hint: The integral on the right hand side has to be evaluated separately, and then plugged into the final solution). [5 points]

ii) $y' = \frac{x^2+1}{y}$, with $y(0) = 0$. Is the solution unique? If not, give all the solutions and determine the interval of existence of both the solutions. [6 points]

iii) $y'' + 4y = 0$, with $y(0) = 1$, $y'(\pi/2) = 1$. (Note: The two intial conditions are at different x). [4 points]

iv) $y' = 1 + 2x + y^2 + 2xy^2$, with $y(0) = 0$. (Hint: $\int \frac{d\theta}{1+\theta^2} = \tan^{-1}\theta$). Determine the interval of existence. The exact numerical values are not required. [5 points]

Problem 2: [10 points] The population, $p(t)$, of some species is governed by the differential equation

$$\frac{dp}{dt} = f(p), \quad \text{where } f(p) = -1 + 2p - p^2; \quad p > 0, t > 0.$$

i) Determine the equilibrium solutions. [2 points]

ii) Plot $f(p)$ versus p and use this to plot the solution curves, i.e. p versus t. Comment on the stability of the equilibrium solution. [5 points]

iii) If p(0) = 1/2, find the time $t_e(> 0)$ at which the population becomes extinct, i.e. $p(t_e) = 0$. [3 points]

Problem 3: [10 points] For the following inhomogeneous differential equation

$$y'' - y' - 2y = 4e^{-t},$$

i) Obtain the general solution. [6 points]

ii) If $y(0) = 0$ and $y'(0) = 1$, obtain the exact solution. Comment on the nature of the solution as $t \to -\infty$ and $t \to \infty$. [4 points]

13|Feb|2012 MATH 215, SECTION 202

MIDTERM I SOLUTIONS

① (i) $\frac{dy}{dx} + y = a \sin x$

$y(0) = 0$

Integrating factor: $\phi = e^{\int dx} = e^x$

$\Rightarrow e^x \frac{dy}{dx} + e^x y = a e^x \sin x$

$\Rightarrow (e^x y)' = a e^x \sin x$

$\Rightarrow e^x y = \int a e^x \sin x \, dx + C$

Let $I = \int e^x \sin x \, dx$

$= \sin x \, e^x - \int \cos x \cdot e^x \, dx = \sin x \, e^x - \left\{ \cos x \, e^x - \int (-\sin x) e^x \, dx \right\}$

$\Rightarrow I = \sin x \cdot e^x - \cos x \, e^x - \underbrace{\int e^x \sin x \, dx}_{I}$

$\Rightarrow 2I = \sin x \, e^x - \cos x \, e^x$

$\Rightarrow I = \frac{1}{2} e^x (\sin x - \cos x)$

$\therefore e^x y = \frac{a}{2} e^x (\sin x - \cos x) + C$

$\Rightarrow y = \frac{a}{2} (\sin x - \cos x) + C e^{-x}$

$y(0) = 0 \Rightarrow 0 = \frac{a}{2}(-1) + C \Rightarrow C = \frac{a}{2}$

$\Rightarrow \boxed{y = \frac{a}{2} \left[\sin x - \cos x + e^{-x} \right]}$

① (ii) $\dfrac{dy}{dx} = \dfrac{x^2+1}{y}$; $y(0) = 0$

In the standard form ; $\dfrac{dy}{dx} = f(x,y)$,

$f(x,y)$ and $\dfrac{\partial f}{\partial y}$ are not continuous at $y=0$

Since $y(0)=0$, we should not be surprized if the solution is not unique.

$y\,dy = (x^2+1)\,dx$

$\Rightarrow \dfrac{y^2}{2} = \dfrac{x^3}{3} + x + C$

$y(0)=0 \Rightarrow C=0$

$\Rightarrow \dfrac{y^2}{2} = \dfrac{x^3}{3} + x \Rightarrow y^2 = 2\left(x + \dfrac{x^3}{3}\right)$

$\Rightarrow y = \pm\sqrt{2\left(x+\dfrac{x^3}{3}\right)}$

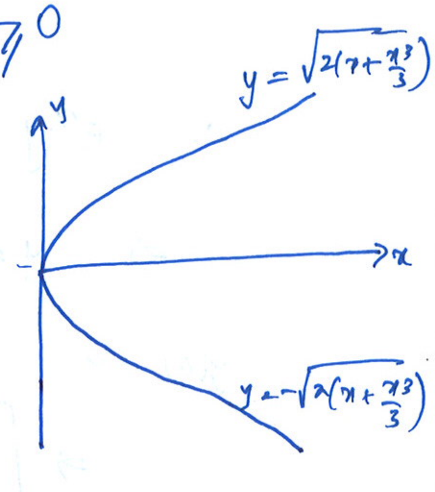
Both solutions satisfy the initial condition. Hence, we dont have a unique solution.

Domain of existence :- For y to be real, we require

$x + \dfrac{x^3}{3} \geq 0$ or; $x\left(1+\dfrac{x^2}{3}\right) \geq 0$

This is always true for $x>0$.

$\therefore$ Domain of existence : $x \geq 0$



1 iii) $y'' + 4y = 0$; $y(0) = 1$; $y'(\frac{\pi}{2}) = 1$

Characteristic equation: $r^2 + 4 = 0$

$\Rightarrow r = \pm 2i$

General Solution: $y = c_1 \cos(2x) + c_2 \sin(2x)$

$y(0) = 1 \Rightarrow 1 = c_1 \cos(0) + c_2 \sin(0)$

$\Rightarrow c_1 = 1$

$y' = -2c_1 \sin(2x) + 2c_2 \cos(2x)$

$\therefore y'(\frac{\pi}{2}) = 1 \Rightarrow -2c_1 \sin(\pi) + 2c_2 \cos(\pi) = 1$

$\Rightarrow -2c_2 = 1 \Rightarrow c_2 = \frac{-1}{2}$

$$\therefore \boxed{y = \cos(2x) - \frac{1}{2}\sin(2x)}$$

---

1 (iv) $y' = 1 + 2x + y^2 + 2xy^2$; $y(0) = 0$

$y' = 1 + 2x + y^2(1+2x)$

$\Rightarrow \frac{dy}{dx} = (1+2x)(1+y^2)$

$\Rightarrow \frac{dy}{1+y^2} = (1+2x)\, dx$

Integrating both sides:

$$\int \frac{dy}{1+y^2} = \int (1+2x)\,dx$$

$$\Rightarrow \tan^{-1}(y) = x + x^2 + C$$

$$\Rightarrow y = \tan(x + x^2 + C)$$

$y(0) = 0 \Rightarrow 0 = \tan(C) \Rightarrow C = n\pi$

But $\tan(n\pi + \theta) = \tan\theta$. Therefore, we let $n = 0$, ie; $C = 0$

$$\Rightarrow y = \tan(x + x^2)$$

Domain of existence:- When $x + x^2 \to \frac{\pi}{2}$; $y \to \infty$

when $x^2 + x - \frac{\pi}{2} = 0$, we get

$$\Rightarrow x = \frac{-1 \pm \sqrt{1 + 4 \cdot \frac{\pi}{2}}}{2}$$

$$= \frac{-1 \pm \sqrt{1 + 2\pi}}{2}$$

$$\boxed{x_{min} = \frac{-1 - \sqrt{1+2\pi}}{2}} \; ; \; \boxed{x_{max} = \frac{-1 + \sqrt{1+2\pi}}{2}}$$

Also, when $x + x^2 \to -\frac{\pi}{2}$; $y \to -\infty$

When $x^2 + x + \frac{\pi}{2} = 0$, we get

$$x = \frac{-1 \pm \sqrt{1 - 4 \cdot \frac{\pi}{2}}}{2} = \frac{-1 \pm \sqrt{1 - 2\pi}}{2}$$

This gives us complex values for x.

$\therefore$ Domain of existence: $x_{min} < x < x_{max}$

Problem 2:

$$\frac{dp}{dt} = f(p) \, ;$$

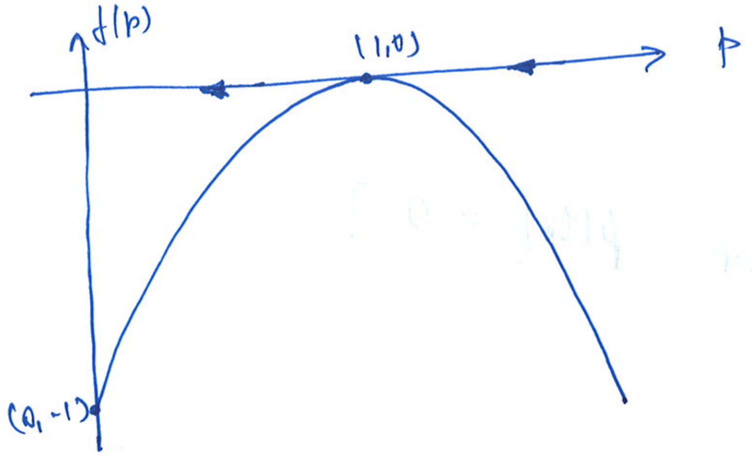
$$f(p) = -1 + 2p - p^2 \qquad ; \quad p > 0 \; ; \quad t > 0$$

(i) Equilibrium Solutions : $f(p) = 0$

$$\Rightarrow \quad -(p-1)^2 = 0$$

$$\Rightarrow \quad p = 1 \qquad \text{(Double root)}$$

(ii)



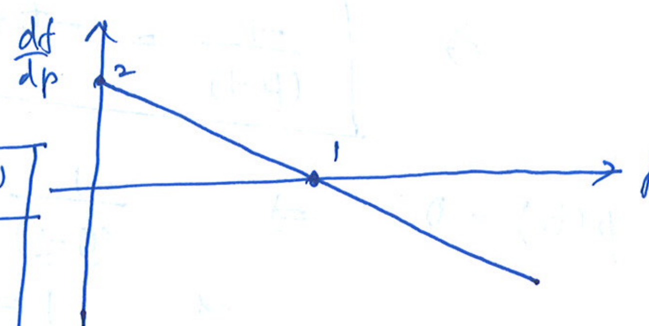
$p = 1$ is a semi-stable equilibrium.

$$\frac{dp}{dt} = f(p)$$

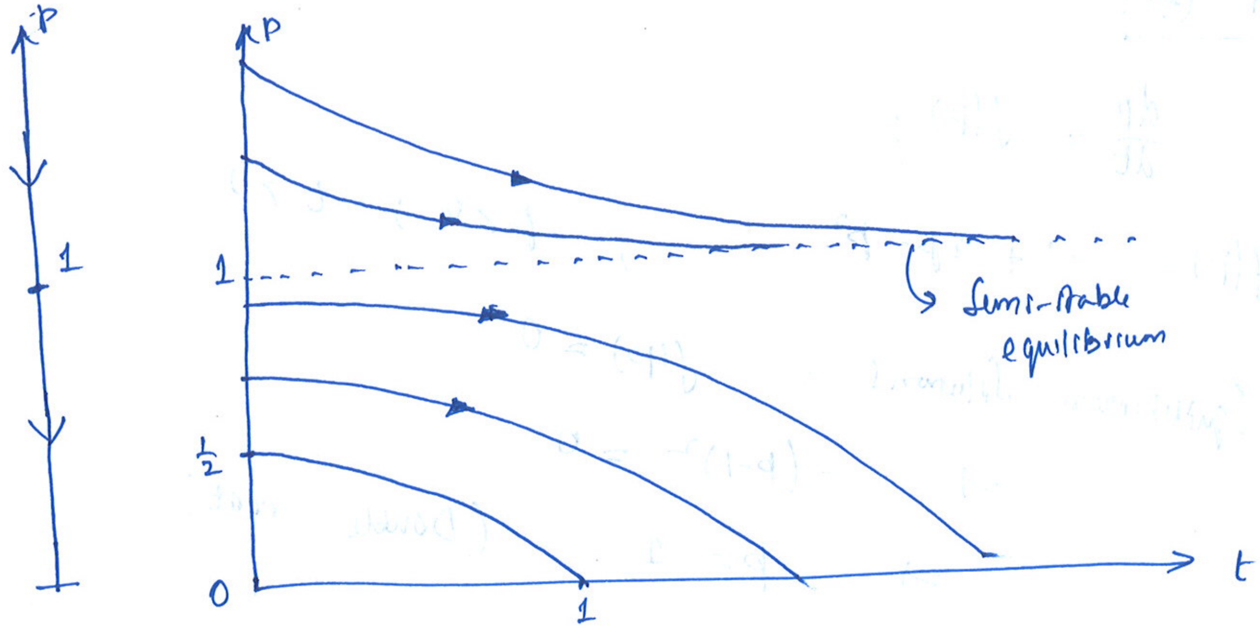
$$\frac{d^2p}{dt^2} = \frac{d}{dt}\left[\frac{dp}{dt}\right] = \frac{d}{dt}\left[f(p)\right] = \frac{d}{dp}\left[f(p)\right] \times \frac{dp}{dt}$$

$$= \frac{df}{dp} \cdot f(p)$$

$$\frac{df}{dp} = 2 - 2p$$



Interval	$\frac{dp}{dt} = f(p)$	$\frac{df}{dp}$	$\frac{d^2p}{dt^2} = \frac{df}{dp} \times f(p)$
$(0,1)$	< 0	> 0	< 0
$(1,\infty)$	< 0	< 0	> 0



(iii) $$\frac{dp}{dt} = -1 + 2p - p^2$$

$$p(0) = \frac{1}{2}$$

Find $t = t_e$ such that $p(t_e) = 0$?

$$\int \frac{dp}{(p-1)^2} = -\int dt$$

$$\Rightarrow \frac{-1}{(p-1)} = -t + C$$

$$p(0) = \frac{1}{2} \Rightarrow \frac{-1}{(\frac{1}{2}-1)} = 0 + C \Rightarrow C = 2$$

$$\Rightarrow \boxed{\frac{-1}{(p-1)} = -t + 2}$$

$$p(t_e) = 0 \Rightarrow \frac{-1}{(0-1)} = -t_e + 2$$

$$\Rightarrow 1 = -t_e + 2 \Rightarrow \boxed{t_e = 1}$$

Problem 3:

$$y'' - y' - 2y = 4e^{-t}$$

(i) Homogeneous part:

$$y'' - y' - 2y = 0$$

Characteristic equation: $r^2 - r - 2 = 0$

$\Rightarrow (r+1)(r-2) = 0$

$\Rightarrow r = -1$ or $r = 2$

$\therefore \; y_1 = e^{-t} \; ; \; y_2 = e^{2t}$

Particular Solution:-

Since the inhomogeneous term $(4e^{-t})$ is same as the solution of the homogeneous equation, we assume a particular solution in the following form:

$$y_p = Ate^{-t}$$

$$y_p' = A\left[-te^{-t} + e^{-t}\right]$$

$$y_p'' = A\left[te^{-t} - e^{-t} - e^{-t}\right]$$

Substituting y_p in the original differential equation, we get

$$A\left[te^{-t} - 2e^{-t}\right] - A\left[-te^{-t} + e^{-t}\right] - 2Ate^{-t} = 4e^{-t}$$

$$\Rightarrow Ae^{-t}\left[-2-1\right] + Ate^{-t}\left[1+1-2\right] = 4e^{-t}$$

$\Rightarrow \quad Ae^{-t}(-3) = 4e^{-t}$

$\Rightarrow \quad -3A = 4 \quad \Rightarrow \quad \boxed{A = \frac{-4}{3}}$

$\therefore \quad y_p = -\frac{4}{3} te^{-t}$

General Solution of the inhomogeneous equation :

$$y = c_1 e^{-t} + c_2 e^{2t} - \frac{4}{3} te^{-t}$$

(ii) $y(0) = 0 \quad ; \quad y'(0) = 1$

$$y'(t) = -c_1 e^{-t} + 2c_2 e^{2t} - \frac{4}{3}\left[e^{-t} - te^{-t}\right]$$

$y(0) = 0 \quad \Rightarrow \quad c_1 + c_2 = 0$

$y'(0) = 1 \quad \Rightarrow \quad -c_1 + 2c_2 - \frac{4}{3}[1 - 0] = 1$

$\Rightarrow \quad c_1 = \frac{-7}{9}$

$\quad\quad c_2 = \frac{7}{9}$

$\therefore \quad \boxed{y = \frac{-7}{9} e^{-t} + \frac{7}{9} e^{2t} - \frac{4}{3} te^{-t}} \rightarrow$ Exact Solution

As $\underline{t \to -\infty}$:

We rewrite the solution as

$$y = -\frac{1}{9}e^{-t}(7+12t) + \frac{7}{9}e^{2t}$$

As $t \to -\infty$; $e^{-t} \to +\infty$

$e^{2t} \to 0$

$7+12t \to -\infty$

$\therefore (7+12t)e^{-t} \to -\infty$

$\therefore y \to +\infty$ as $t \to -\infty$

As $\underline{t \to +\infty}$:

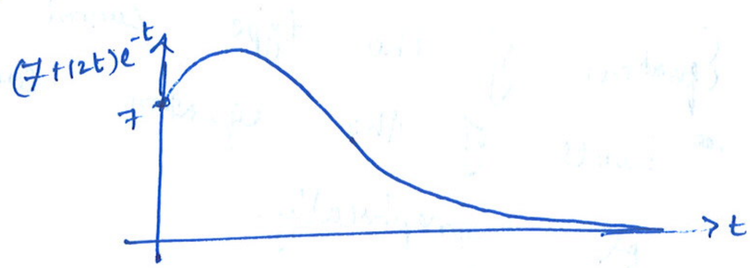
$e^{-t} \to 0$

$e^{2t} \to \infty$

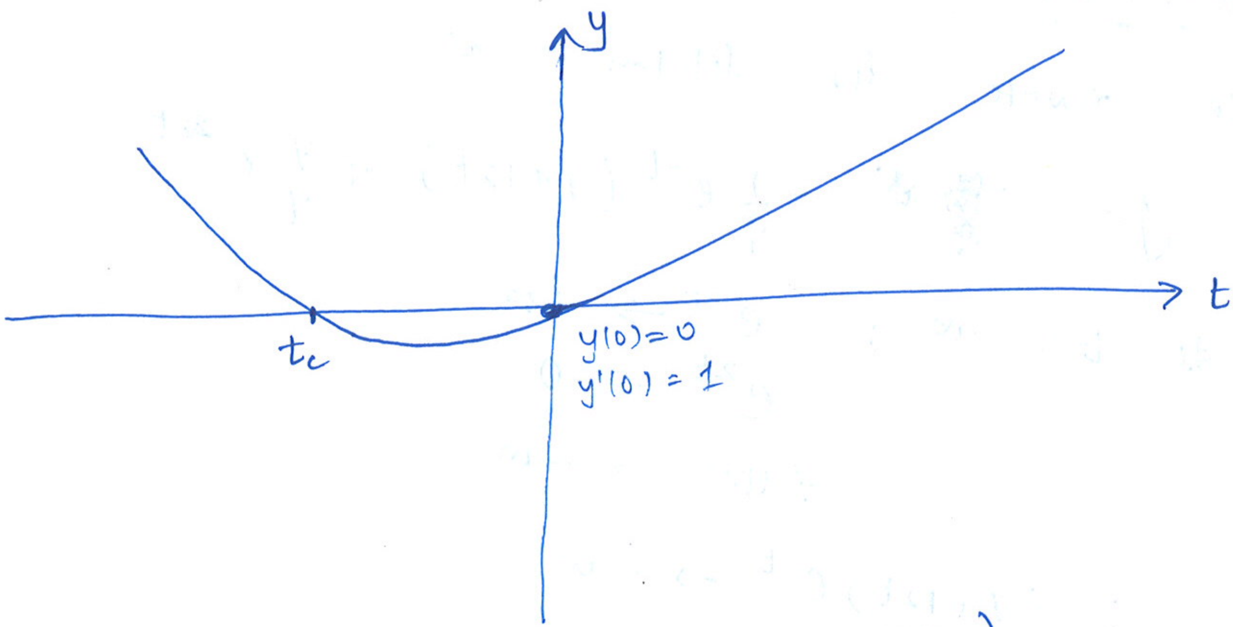
$7+12t \to +\infty$

To determine the limit of $(7+12t)e^{-t}$, we use L'Hospital's rule:

$$\lim_{t \to +\infty} (7+12t)e^{-t} = \lim_{t \to +\infty} \frac{(7+12t)}{e^t} = \lim_{t \to \infty} \frac{12}{e^t} \to 0$$



$\therefore y \to +\infty$



To determine t_c: (Not part of the exam)

$y=0$ at $t=t_c$

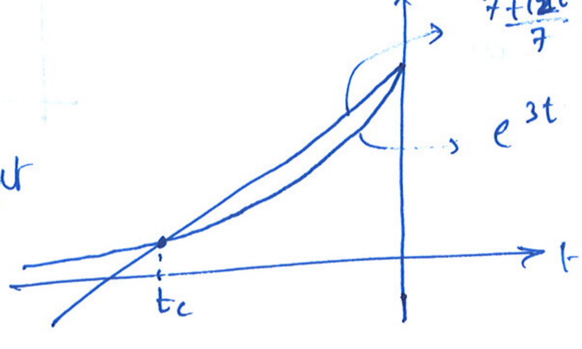
$$\Rightarrow \quad 0 = -\frac{1}{9}e^{-t_c}(7+12t_c) + \frac{7}{9}e^{2t_c}$$

$$\Rightarrow \quad e^{-t_c}(7+12t_c) = 7e^{2t_c}$$

$$\Rightarrow \quad 7e^{3t_c} = 7+12t_c$$

$$\Rightarrow \quad \boxed{e^{3t_c} = \frac{7+12t_c}{7}} \rightarrow \text{TRANSCENDENTAL RELATIONS}$$

Equations of this type cannot be solved in the usual sense. Roots of this equation can only be found numerically or graphically.



In this case, we get

$t_c = -0.4157$