2|Mar|2012

Ex: Find inverse transform of

$$e^{-cs}F(s)$$

$$F(s) = \frac{1-e^{-2s}}{s^2}$$

$$f(t) = \mathcal{L}^{-1}\{F(s)\} = \mathcal{L}^{-1}\left\{\frac{1}{s^2}\right\} - \mathcal{L}^{-1}\left\{\frac{e^{-2s}}{s^2}\right\}$$

$$= t - u_2(t)\,(t-2)$$

$$f(t) = \begin{cases} t & 0 \le t < 2 \\ 2 & t \geq 2 \end{cases}$$

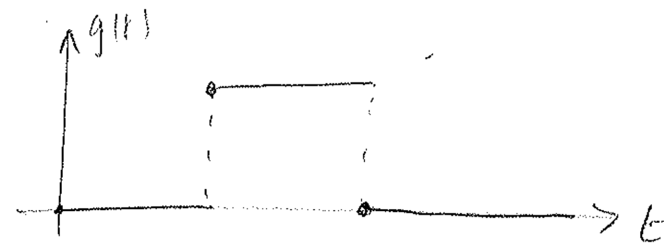
Differential Equations with Discontinuous Forcing :-

Ex: $y'' + y' + y = g(t)$ $= \begin{cases} 1 & 1 \le t < 2 \\ 0 & 0 \le t < 1 \text{ and } t \geq 2 \end{cases}$

$g(t) = u_1(t) - u_2(t)$

$y(0) = 0$

$y'(0) = 0$



(A)

$$[s^2Y(s) - sy(0) - y'(0)] + [sY(s) - y(0)] + Y(s) = \mathcal{L}\{u_1(t)\} - \mathcal{L}\{u_2(t)\}$$

(B) $$(s^2+s+2)\,Y(s) = \frac{e^{-s}}{s} - \frac{e^{-2s}}{s}$$

$$\Rightarrow \quad Y(s) = \frac{e^{-s} - e^{-2s}}{s(s^2+s+1)} = (e^{-s} - e^{-2s})\,H(s)$$

When $H(s) = \dfrac{1}{s(s^2+s+1)}$

$$y(t) = u_1(t)\, h(t-1) - u_2(t)\, h(t-2)$$

$$H(s) = \frac{1}{s(s^2+s+1)} = \frac{a}{s} + \frac{bs+c}{s^2+s+1}$$

$$= \frac{(a+b)s^2 + (a+c)s + a}{s(s^2+s+1)}$$

$$\begin{aligned} a+b &= 0 \\ a+c &= 0 \\ a &= 1 \end{aligned} \Rightarrow \begin{aligned} a &= 1 \\ b &= -1 \\ c &= -1 \end{aligned}$$

$$s^2+s+1 = s^2+2\cdot s\cdot\tfrac{1}{2}+\tfrac{1}{4}-\tfrac{1}{4}+1 = \left(s+\tfrac{1}{2}\right)^2+\tfrac{3}{4}$$

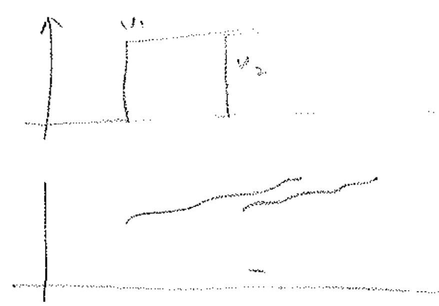
$$H(s) = \frac{1}{s} - \frac{s+1}{s^2+s+1} = \frac{1}{s} - \frac{s+1}{\left(s+\frac{1}{2}\right)^2+\left(\frac{\sqrt{3}}{2}\right)^2}$$

$$= \frac{1}{s} - \frac{\left(s+\frac{1}{2}\right)}{\left(s+\frac{1}{2}\right)^2+\left(\frac{\sqrt{3}}{2}\right)^2} - \frac{1}{\sqrt{3}}\frac{\sqrt{3}/2}{\left(s+\frac{1}{2}\right)^2+\left(\frac{\sqrt{3}}{2}\right)^2}$$

$$h(t) = 1 - e^{-\frac{1}{2}t}\cos\left(\frac{\sqrt{3}}{2}t\right) - \frac{1}{\sqrt{3}}e^{-\frac{1}{2}t}\sin\left(\frac{\sqrt{3}}{2}t\right)$$

: Damped Oscillation

$$\Rightarrow\ y(t) = u_1(t)\, h(t-1) - u_2(t)\, h(t-2)$$



10. $Y(s)$

For $t<1$:

$$y(t) = 0$$

For $1 \leqslant t < 2$:

$$y(t) = 1 - e^{-\frac{1}{2}(t-1)} \cos\left[\frac{\sqrt{3}}{2}(t-1)\right] - \frac{1}{\sqrt{3}} e^{-\frac{1}{2}(t-1)} \sin\left[\frac{\sqrt{3}}{2}(t-1)\right]$$

$$y'(t) = \frac{1}{2} e^{-\frac{1}{2}(t-1)} \cos\left(\frac{\sqrt{3}}{2}(t-1)\right) + e^{-\frac{1}{2}(t-1)} \sin\left(\frac{\sqrt{3}}{2}(t-1)\right)\left(\frac{\sqrt{3}}{2}\right)$$

$$- \frac{1}{\sqrt{3}}\left(\frac{-1}{2}\right) e^{-\frac{1}{2}(t-1)} \sin\left(\frac{\sqrt{3}}{2}(t-1)\right) - \frac{1}{\sqrt{3}} e^{-\frac{1}{2}(t-1)} \cos\left[\frac{\sqrt{3}}{2}(t-1)\right]\frac{\sqrt{3}}{2}$$

$$y''(t) = \frac{1}{2} e^{-\frac{1}{2}(t-1)}\left(\frac{-1}{2}\right) \cos\left[\frac{\sqrt{3}}{2}(t-1)\right) - \frac{1}{2} e^{-\frac{1}{2}(t-1)} \sin\left(\frac{\sqrt{3}}{2}(t-1)\right]\frac{\sqrt{3}}{2}$$

$$+ e^{(\ )} \sin(\ ) + e^{-\frac{1}{2}(t-1)} \frac{\sqrt{3}}{2} \cdot \frac{\sqrt{3}}{2} \cos\left[\frac{\sqrt{3}}{2}(t-1)\right]$$

$$+ \frac{1}{2\sqrt{3}} e^{(\ )} \sin(\ ) + \frac{1}{2\sqrt{3}} e^{-\frac{1}{2}(t-1)} \frac{\sqrt{3}}{2} \cos\left[\frac{\sqrt{3}}{2}(t-1)\right]$$

$$- \frac{\sqrt{3}}{2}\frac{1}{\sqrt{3}}\left(\frac{-1}{2}\right) e^{-\frac{1}{2}(t-1)} \cos\left[\frac{\sqrt{3}}{2}(t-1)\right] - \frac{1}{\sqrt{3}} e^{(\ )} \sin(\ )$$

$$= \frac{-1}{4} + \frac{3}{4} + \frac{1}{4} + \frac{1}{4} = 1$$

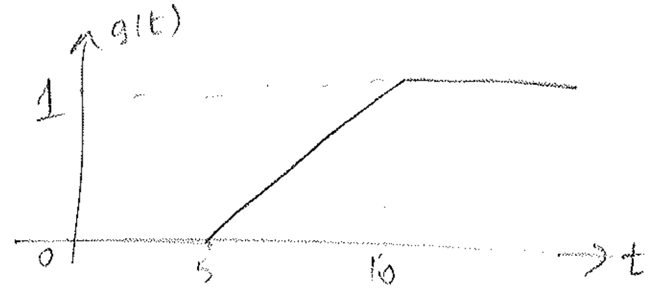
$$\therefore \lim_{t \to 1^-} y''(t) = 0 \quad ; \quad \lim_{t \to 1^+} y''(t) = 1$$

Ex:

$$y'' + 4y = g(t)$$

$$y(0) = 0 \quad ; \quad y'(0) = 0$$

$$g(t) = \begin{cases} 0 & 0 \le t < 5 \\ (t-5)/5 & 5 \le t < 10 \\ 1 & t \geq 10 \end{cases}$$



for $t < 5$: $\quad y'' + 4y = 0 \Rightarrow r^2 + 4 = 0$

$\Rightarrow r = \pm 2i$

$$y = c_1 \cos(2t) + c_2 \sin(2t)$$

for $t \geq 10$: $\quad y = c_1 \cos(2t) + c_2 \sin(2t) + \frac{1}{4}$

$y'' + 4y = 1$

$$g(t) = u_5(t)\left(\frac{t-5}{5}\right) - u_{10}(t)\left(\frac{t-10}{5}\right)$$

$$\cancel{s^2 F(s)}\ s^2 Y(s) + 4Y(s) = \frac{1}{5}\left[\frac{e^{-5s}}{s^2} - \frac{e^{-10s}}{s^2}\right]$$

$$\Rightarrow Y(s) = \frac{e^{-5s} - e^{-10s}}{5s^2(s^2+4)} = \frac{1}{5}\left[e^{-5s} - e^{-10s}\right] H(s)$$

where $\quad H(s) = \dfrac{1}{s^2(s^2+4)} = \dfrac{1/4}{s^2} - \dfrac{1/4}{s^2+4}$

$$h(t) = \frac{1}{4}t - \frac{1}{4} \cdot \frac{1}{2} \sin(2t) = \frac{t}{4} - \frac{1}{8} \sin 2t$$

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$\Rightarrow$ $\cancel{a}$ $y(t) = \frac{1}{5}\left[u_5(t)\, h(t-5) - u_{10}(t)\, h(t-10) \right]$

where $h(t) = \frac{t}{4} - \frac{1}{8}\sin(2t)$