17/2/2012



B **Ex:**

$$f(t) = \begin{cases} 1 & 0 \le t < 1 \\ k & t = 1 \\ 0 & t > 1 \end{cases} \qquad k = \text{constant}$$

$$\mathcal{L}\{f(t)\} = \int_0^\infty f(t) e^{-st}\,dt = \int_0^1 f(t) e^{-st}\,dt + \int_1^\infty f(t) e^{-st}\,dt$$

$$= \int_0^1 e^{-st}\,dt + 0$$

$$= \left. \frac{e^{-st}}{-s} \right]_0^1 = \frac{1-e^{-s}}{s} \; ; \; s > 0$$

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$$\mathcal{L}\{f(t)\} = F(s) = \int_0^\infty e^{-st} f(t)\,dt$$

$$\mathcal{L}\{1\} = \frac{1}{s} \; ; \; s > 0$$

$$\mathcal{L}\{e^{at}\} = \frac{1}{s-a} \; ; \; s > a$$

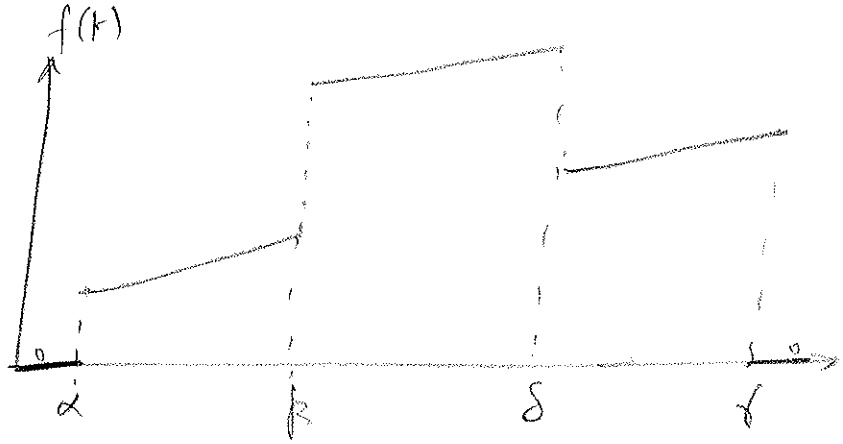
$$\mathcal{L}\{e^{at} f(t)\} = F(s-a)$$

$$\mathcal{L}\{t^n\} ; n \text{ positive Integer} = \frac{n!}{s^{n+1}} \; ; \; s > 0$$

$$\mathcal{L}\{\cos at\} = \frac{s}{s^2+a^2} \; ; \; s > 0$$

$$\mathcal{L}\{\sin at\} = \frac{a}{s^2+a^2} \; ; \; s > 0$$

Piece wise continuous function:




$$\int_0^\infty f(t)\,dt$$

$$= \int_0^\alpha \cancel{f(t)\,dt} + \int_\alpha^\beta f(t)\,dt + \int_\beta^\delta f(t)\,dt + \int_\gamma^\infty \cancel{f(t)\,dt}$$

The value at $\alpha, \beta, \delta, \gamma$ are unimportant.

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Convergence of Laplace Transform:

$f(t)$ ~~should~~ can only grow as rapidly as

$e^{-st} f(t)$ can converge.

How fast can $f(t)$ grow?

$f(t)$ should be of exponential type.

$$|f(t)| \leq Ke^{at} \quad \text{for } t \geq 0$$

$K > 0$ : constant.

$a > 0$ : constant

Let $f(t) = \cos t$

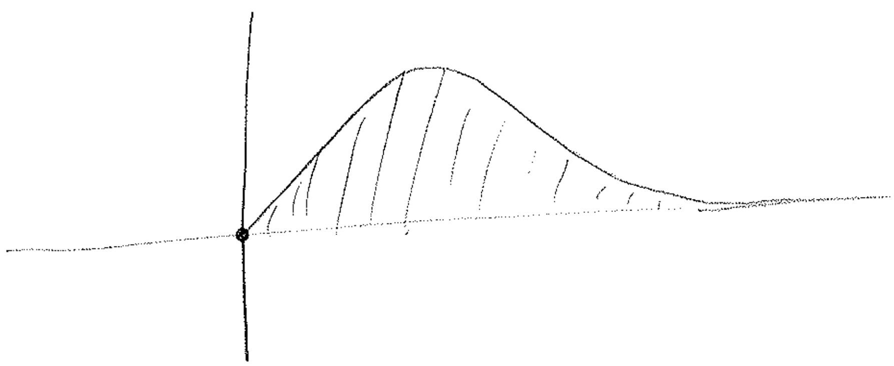
$\longrightarrow$ $|\cos t| \leq 1 \cdot e^{0 \cdot t}$ $\quad K = 1$, $a = 0$

$\therefore \cos t$ is of exponential type.

$\longrightarrow$ $|t^n| \leq \underline{Ke^t}$ for some $K$

$K\left(1 + t + \frac{t^2}{2!} + \cdots + \frac{t^n}{n!} + \frac{t^{n+1}}{(n+1)!} + \cdots\right)$

$\frac{|t^n|}{e^t} \leq K$ : Is this true as $t \to \infty$. $\lim_{t \to \infty}\left(\frac{t^n}{e^t}\right)$ L'Hôpital's rule $n$ times



$\longrightarrow$ $f(t) = \frac{1}{t}$ :

$\frac{1}{t}$ : $\int_0^\infty \frac{1}{t} e^{-st}\,dt$ When $t \to 0$; the integral diverges.

$\longrightarrow$ $f(t) = e^{t^2}$

If $e^{t^2} \leq Ke^{at}$ $\quad$ ~~$\frac{e^{t^2}}{e^{at}} \leq K$~~ $\quad t^2 = at$

After $t > a$, $e^{t^2} > e^{at}$

Derivative formula :-

Let $|f(t)| \leq Ke^{at}$

$$\mathcal{L}\{f'(t)\} = ?$$

$$= \int_0^\infty f'(t)e^{-st}\,dt = \lim_{R\to\infty}\int_0^R e^{-st}f'(t)\,dt$$

$$\int_0^R e^{-st}f'(t)\,dt = \left[e^{-st}f(t)\right]_0^R - \int_0^R -se^{-st} f(t)\,dt$$

$$= \underbrace{\left\{e^{-sR}f(R) - f(0)\right\}} + s\int_0^R e^{-st}f(t)\,dt$$

As $R \to \infty$ $\quad e^{-sR}f(R) \longrightarrow 0$

Since $f(t) \leq Ke^{at}$

$e^{-sR} \times Ke^{aR}$

$= Ke^{-(s-a)R}$

$\lim_{R\to\infty} f(R)e^{-sR} \to 0$ for $s > a$

$$\Rightarrow \mathcal{L}\{f'(t)\} = -f(0) + s\,\mathcal{L}\{f(t)\}$$

If $f(t) = y(t)$ :

$$\mathcal{L}\{y'\} = s\,\mathcal{L}\{y\} - y(0)$$

$\mathcal{L}\{y''\} = \mathcal{L}\{(y')'\}$

let $y' = f$

$= \mathcal{L}\{f'\}$

$= s\,\mathcal{L}\{f\} - f(0)$

$= s\mathcal{L}\{y'\} - y'(0)$

$= s\left[s\mathcal{L}\{y\} - y(0)\right] - y'(0)$

$= s^2\mathcal{L}\{y\} - s\,y(0) - y'(0)$

Let $\mathcal{L}\{y\} = Y$ : Notation.

Solving a D.E.

$$\text{IVP}\begin{cases} y'' + py' + qy = g \\ y(0) = y_0 \; ; \quad y'(0) = y_0' \end{cases}$$

Step 1: Take Laplace transform of the equation.

$\mathcal{L}\{y'' + py' + qy\} = \mathcal{L}\{g\}$

Step 2: Get an equation for $Y(s)$

Typically ; $Y(s) = \frac{A(s)}{B(s)}$

Simplify using partial fractions.

Step 4: Calculate $y(s) = \mathcal{L}^{-1}\{Y(s)\}$

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Ex: (i) $y'' - y' - 2y = 0$

$y(0) = 1 \quad ; \quad y'(0) = 0$

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Step 1: $\mathcal{L}\{y'' - y' - 2y\} = 0$

$\mathcal{L}\{y''\} - \mathcal{L}\{y'\} - 2\mathcal{L}\{y\} = 0$

Let $\mathcal{L}\{y\} = Y(s)$

Step 2: $\{s^2 \mathcal{L}\{y\} - s y(0) - y'(0)] - [s\mathcal{L}\{y\} - y(0)\}$

$- 2\mathcal{L}\{y\} = 0$

$\Rightarrow \cancel{s^2 Y(s)} \; (s^2 - s - 2)\, Y(s) + (1-s)\underbrace{y(0)}_{1} - \underbrace{y'(0)}_{0} = 0$

$\Rightarrow (s^2 - s - 2)\, Y(s) + (1-s) = 0$

$\Rightarrow Y(s) = \frac{s-1}{s^2 - s - 2} = \frac{s-1}{(s-2)(s+1)}$

$$Y(s) = \frac{(1/3)}{s-2} + \frac{(2/3)}{s+1}$$

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Step:

$$y(t) = \mathcal{L}^{-1}\{Y(s)\}$$

$$= \mathcal{L}^{-1}\left\{\frac{1}{3}\frac{1}{(s-2)}\right\} + \mathcal{L}^{-1}\left\{\frac{2}{3}\frac{1}{s+1}\right\}$$

$$\mathcal{L}\{e^{at}\} = \frac{1}{s-a} \Rightarrow \mathcal{L}^{-1}\left\{\frac{1}{s-a}\right\} = e^{at}$$

$$= \frac{1}{3}e^{2t} + \frac{2}{3}e^{-t}$$

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Ex:

$$y'' + y = \sin(2t)$$

$$y(0) = 2 \,; \quad y'(0) = 1$$

$$\mathcal{L}\{\sin at\} = \frac{a}{s^2+a^2}$$

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$$\left[s^2 Y(s) - s\,y(0) - y'(0)\right] + Y(s) = \frac{2}{s^2+4}$$

$$\Rightarrow s^2 Y(s) - 2s - 1 + Y(s) = \frac{2}{s^2+4}$$

$$\Rightarrow (s^2+1)\,Y(s) = \frac{2}{s^2+4} + 1 + 2s$$

$$= \frac{2s^3 + 8s + s^2 + 4 + 2}{s^2+4}$$

$$\Rightarrow \quad Y(s) = \frac{2s^3 + s^2 + 8s + 6}{(s^2+1)(s^2+4)}$$

$$= \frac{as+b}{s^2+1} + \frac{cs+d}{s^2+4}$$

$a = 2$; $c = 0$; $b = \frac{5}{3}$; $d = -\frac{2}{3}$

$$Y(s) = \frac{2s + 5/3}{(s^2+1)} + \frac{(-2/3)}{s^2+4}$$

$$= \frac{2s}{s^2+1} + \frac{5/3}{s^2+1} - \frac{2/3}{s^2+4}$$

$$= 2\cos(t) + \frac{5}{3}\sin t - \frac{1}{3}\sin(2t)$$

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$$\mathcal{L}\{f^{n}(t)\} = s^{n}\mathcal{L}\{f(t)\} - s^{n-1}f(0) - \cdots - sf^{n-2}(0) - f^{(n-1)}(0)$$