THE LAPLACE TRANSFORM

Laplace transform is one of the many integral transforms used to solve differential equation. But what is an integral transform, or more generally a transform?

Operator vs Transform :-

Recall that an operator takes a function $y(t)$ and gives us a new function of t, often involving derivatives of $y(t)$.

Ex: $L[y] = y'' + ay' + by$ ∴ The output is still in terms of t.

A transform takes in a function of t, but outputs a function of a different variable, say s.

$$f(t) \longrightarrow \boxed{\text{Operator}} \longrightarrow g(t)$$

$$f(t) \longrightarrow \boxed{\text{TRANSFORM}} \longrightarrow F(s)$$

Why use a Laplace Transform?

(i) Using a Laplace transform, we can eliminate derivatives thus making it into a algebraic problem.

(ii) The nonhomogeneous term can be more "complicated" such as discontinuous functions.

Origin of Laplace Transform :- (Can be skipped if you like)

Lets recall a standard power series :

Consider $\sum_{n=0}^{\infty} a_n x^n = A(x)$

We sum over all n from 0 to ∞. So what's left is a function of x.

Ex: If $a_n = \frac{1}{n!}$, we have

$$\sum_{n=0}^{\infty} \frac{1}{n!} x^n = 1 + x + \frac{x^2}{2!} + \cdots = \underbrace{e^x}_{A(x)}$$

We can rewrite the power series as

$$\sum_{n=0}^{\infty} a(n)\, x^n = A(x)$$

where $a(n)$ is now a function of n.

What the power series is really doing is the following: for a function $a(n)$, we have an associated function $A(x)$.

$$a(n) \longrightarrow A(x)$$

In the above series, n assumes only integer values.

We can "convert" the summation to an integral:

for $n = 0, 1, 2, \ldots$, we use $t \in [0, \infty)$.

So we have, with $n \to t$,

$$\underbrace{\int_0^\infty a(t)\, x^t \, dt}_{\text{we are integrating } t \text{ out}} = A(x)$$

Note that this is an improper integral (discussed later).

for the integral to converge, we "require" $0 < x < 1$.

~~LHB~~ since $x = e^{\ln x}$,

$$x^t = (e^{\ln x})^t$$

since $0 < x < 1$, $\ln x < 0$. Define $s = -\ln x$, we get

$$\boxed{\int_0^\infty a(t)\, e^{-st}\, dt = A(s)}$$

$(\because\ x = e^{-s})$

$\Rightarrow A(x) = A(s)$

<u>Definition</u>: <u>Laplace Transform</u>: Let $f(t)$ be a "well-defined" function for $t \geq 0$. The Laplace transform of $f(t)$, denoted by $\mathcal{L}\{f(t)\}$ or by $F(s)$ is defined as

$$\boxed{\mathcal{L}\{f(t)\} = F(s) = \int_0^\infty e^{-st} f(t)\, dt}$$

This is an improper integral. Therefore the Laplace transform is only defined if the integral converges.

Theorem 6.1.2:

Convergence of Laplace Transform

$f(t)$ is allowed to grow only as rapidly as

$e^{-st} \cdot f(t)$ converges.

This happens when $f(t)$ is of exponential order/type,

i.e; $|f(t)| \leq K e^{at}$ for $t \geq M$,

with $K > 0$: constant

$a > 0$: constant.

M is a positive constant.

Note that if $f(t)$ is a piece-wise continuous function defined on an interval $t \in [0, A]$, the integral automatically converges.

Proof: $\int_0^\infty e^{-st} f(t)\, dt = \int_0^M e^{-st} f(t)\, dt + \int_M^\infty e^{-st} f(t)\, dt$

Convergent if $f(t)$ is piecewise continuous.

If $|f(t)| \leq Ke^{at}$ for $t \geq M$, then

$$\int_M^\infty e^{-st} f(t)\, dt \quad \text{converges for } s>a \text{ since}$$

$$|e^{-st} f(t)| \le K e^{-st} e^{at} = K e^{-(s-a)t}$$

for $s>a$, $e^{-(s-a)t} \to 0$ as $t \to \infty$.

Hence proved.

Summary: The Laplace transform $\mathcal{L}\{f(t)\} = F(s) = \int_0^\infty e^{-st} f(t)\, dt$ is well defined for $s>a$.

Some basics: 1 Improper integral

When dealing with Laplace transforms, we encounter a lot of improper integrals. An improper integral over an unbounded interval is defined as limit of integrals over finite intervals, ie;

$$\int_a^\infty f(t)\, dt = \lim_{R\to\infty} \int_a^R f(t)\, dt$$

where R is a positive real number.

If the integral from a to R exists for each $R>a$, and if the limit $R\to\infty$ exists, then the improper integral is said to converge.

Ex: If $f(t) = e^{ct}$, $t \geq 0$, then

$$\int_0^\infty e^{ct}\,dt = \lim_{R\to\infty} \int_0^R e^{ct}\,dt = \lim_{R\to\infty} \left.\frac{e^{ct}}{c}\right]_0^R$$

$$= \lim_{R\to\infty} \frac{1}{c}\left(e^{cR}-1\right)$$

If $c < 0$, then $\lim_{R\to\infty} e^{cR} \to 0 \Rightarrow \int_0^\infty e^{ct}\,dt = \frac{-1}{c}$

If $c > 0$, then $\lim_{R\to\infty} e^{cR} = \infty \Rightarrow \int_0^\infty e^{ct}\,dt$ diverges.

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(2) **Piecewise Continuous functions:-** A function $f(t)$ is said to be piecewise continuous on an interval $t \in [\alpha, \beta]$ if the interval can be partitioned by a finite number of points $\alpha = t_0 < t_1 < \cdots < t_n = \beta$ with

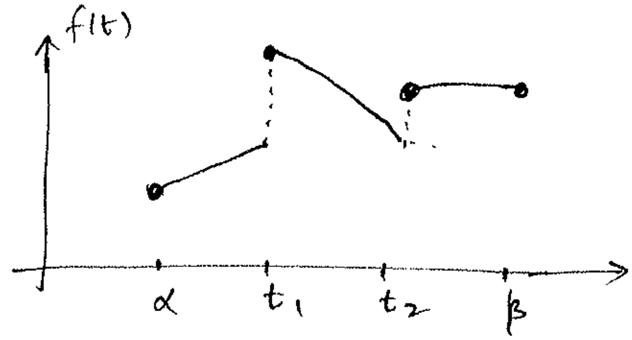
(i) $f(t)$ continuous on each interval $t_{i-1} < t < t_i$;

(ii) $\lim_{t\to t_{i-1}} f(t)$ and $\lim_{t\to t_i} f(t)$ exist and are finite for each i.

If f is piecewise continuous on $t \in [\alpha, \beta]$ for every $\beta > \alpha$, then f is said to be piecewise continuous on $t \in [\alpha, \infty)$.

For a piecewise continuous function defined on $[\alpha, \beta]$ with jumps at t_1 and t_2, we have

$$\int_\alpha^\beta f(t)\,dt = \int_\alpha^{t_1} f(t)\,dt + \int_{t_1}^{t_2} f(t)\,dt + \int_{t_2}^{\beta} f(t)\,dt.$$



Ex: For $f(t) = \begin{cases} 0 & 0 < t < 1 \\ t & 1 \leqslant t < 3 \\ t^2 & 3 \leqslant t < 4 \\ 0 & t \geqslant 4 \end{cases}$.

Calculate $\int_0^\infty f(t)\,dt$.

$$\int_0^\infty f(t)\,dt = \int_0^1 f(t)\,dt + \int_1^3 f(t)\,dt + \int_3^4 f(t)\,dt + \int_4^\infty f(t)\,dt$$

$$= 0 + \int_1^3 t\,dt + \int_3^4 t^2\,dt + 0$$

$$= \left.\frac{t^2}{2}\right]_1^3 + \left.\frac{t^3}{3}\right]_3^4 = \frac{1}{2}(9-1) + \frac{1}{3}(64-27)$$

$$= 4 + \frac{37}{3} = \frac{49}{3} \quad \#$$

Note that the function value at the jumps is unimportant.

(3) Exponential Order: If $|f(t)| \leq K e^{at}$ for $t \geq 0$, with $K > 0$ & $a \geq 0$, then $f(t)$ is said to be of Exponential type/order.

Ex: (i) $f(t) = \cos t$.

We need to find a K such that the inequality is satisfied.

$$|\cos t| \leq 1 \cdot e^{0 \cdot t} \quad \text{where } K = 1,\ a = 0$$

The inequality is true $\Rightarrow$ $\cos t$ is of exponential order.

Ex (ii) $f(t) = t^n$

$$|t^n| \leq K e^{at}$$

If $a = 1$, then $K e^{at} = K\left(1 + t + t^2 + \cdots + \frac{t^n}{n!} + \frac{t^{n+1}}{(n+1)!} + \cdots\right)$

for any K, ~~we~~ ~~can~~ the inequality is always true. This can be seen by calculating

$\lim_{t \to \infty} \left(\frac{t^n}{e^t}\right)$. Using L'Hopital's rule, this goes to zero..

$$\Rightarrow \frac{|t^n|}{e^t} \leq K \Rightarrow |t^n| \text{ is of exponential order.}$$

~~Ex: (iii) $f(t) = \frac{1}{t}$~~

Laplace Transforms : Examples

Before proceeding to solve differential equations, let us first calculate the Laplace transforms of certain sample functions.

Ex (i): $f(t) = 1$. Calculate $F(s)$.

$$F(s) = \mathcal{L}\{f(t)\} = \int_0^\infty f(t)\, e^{-st}\, dt$$

$$= \int_0^\infty 1 \cdot e^{-st}\, dt = \lim_{R\to\infty} \int_0^R e^{-st}\, dt$$

$$= \lim_{R\to\infty} \left.\frac{e^{-st}}{-s}\right|_0^R = \lim_{R\to\infty} \frac{e^{-sR}-1}{-s}$$

If $s>0$, $e^{-sR} \to 0$ as $R\to\infty$.

$$\Rightarrow \quad F(s) = \frac{1}{s} \quad \text{if } s>0.$$

Ex (ii) $f(t) = e^{at}$, $t \geq 0$.

$$F(s) = \mathcal{L}\{e^{at}\} = \int_0^\infty e^{at}\cdot e^{-st}\, dt = \int_0^\infty e^{-(s-a)t}\, dt$$

$$= \lim_{R\to\infty} \int_0^R e^{-(s-a)t}\, dt = \lim_{R\to\infty} \left.\frac{e^{-(s-a)t}}{-(s-a)}\right|_0^R$$

$$= \lim_{R\to\infty} \frac{e^{-(s-a)R}-1}{-(s-a)} = \frac{1}{s-a} \quad \text{if } s>a.$$

Therefore $\mathcal{L}\{e^{at}\} = \frac{1}{s-a}$, $s > a$

Ex (iii) The previous example can also be extended to complex exponentials.

$$f(t) = e^{(a+ib)t}$$

$$F(s) = \mathcal{L}\{e^{(a+ib)t}\} = \frac{1}{s-(a+ib)}, \quad s > a$$

$$= \frac{1}{(s-a) - ib} = \frac{1}{(s-a)-ib} \times \frac{(s-a)+ib}{(s-a)+ib}$$

$$= \frac{(s-a)+ib}{(s-a)^2+b^2}$$

If $b=0$, we recover result from Ex(ii).

If $a=0$, $f(t) = e^{ibt}$

$$\Rightarrow \mathcal{L}\{e^{ibt}\} = \frac{s+ib}{s^2+b^2}$$

Since $e^{ibt} = \cos(bt) + i\sin(bt)$, we have

$$\mathcal{L}\{\cos bt + i\sin bt\} = \frac{s+ib}{s^2+b^2}$$

Def! If f_1 and f_2 are two functions, then

$$\mathcal{L}\{c_1 f_1 + c_2 f_2\} = c_1\mathcal{L}\{f_1\} + c_2\mathcal{L}\{f_2\}$$

ie; Laplace Transform is a Linear Operator.

Using linearity of Laplace transform, we have

$$\mathcal{L}\{\cos bt + i\sin bt\} = \mathcal{L}\{\cos bt\} + i\,\mathcal{L}\{\sin bt\} = \frac{s+ib}{s^2+b^2}$$

Comparing real and imaginary parts, we have

$$\mathcal{L}\{\cos bt\} = \frac{s}{s^2+b^2}, \quad s>0$$

$$\mathcal{L}\{\sin bt\} = \frac{b}{s^2+b^2}, \quad s>0$$

Ex (iv): Verify result of Ex (iii) by direct integration.

$$\mathcal{L}\{\sin at\} = \int_0^\infty \sin(at)\,e^{-st}\,dt \,. \quad \cancel{s>0}$$

Using integration by parts,

$$F(s) = \mathcal{L}\{\sin at\} = \lim_{R\to\infty}\left[e^{-st}\cdot\frac{\cos(at)}{-a}\Big|_0^R - \int_0^R (-s)e^{-st}\cdot\frac{\cos(at)}{(-a)}\,dt \right]$$

$$= \lim_{R\to\infty}\left[-\frac{1}{a}\left(e^{-sR}\cos(aR)-1\right)\right] - \frac{s}{a}\int_0^R e^{-st}\cos(at)\,dt$$

$$= \frac{1}{a} - \lim_{R\to\infty}\left[\frac{1}{a}e^{-sR}\cos(aR)\right] - \frac{s}{a}\left\{ e^{-st}\,\frac{\sin(at)}{a}\Big|_0^R - \int_0^R (-s)e^{-st}\,\frac{\sin(at)}{a}\,dt \right\}$$

If $s>0$, then $\lim_{R\to\infty} e^{-sR}\cos(aR) = 0$

with $s > 0$, we have

$$\mathcal{L}\{\sin(at)\} = \frac{1}{a} - \frac{s^2}{a^2} \underbrace{\int_0^\infty e^{-st} \sin(at)\, dt}_{\mathcal{L}\{\sin at\}}$$

$$\Rightarrow \mathcal{L}\{\sin(at)\} \cdot \left[1 + \frac{s^2}{a^2}\right] = \frac{1}{a}$$

$$\Rightarrow \mathcal{L}\{\sin(at)\} = \frac{a}{s^2+a^2}, \quad \text{with } s > 0.$$

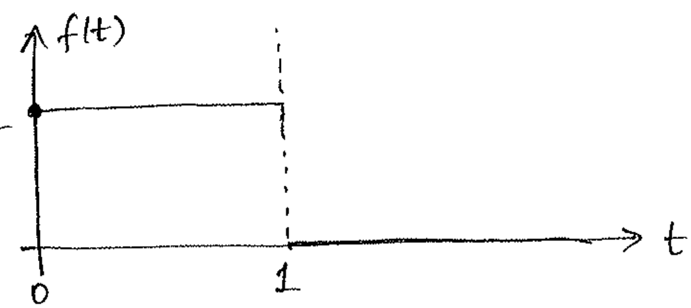
We can use the same procedure to verify that

$$\mathcal{L}\{\sin(at)\} = \frac{s}{s^2+a^2}, \quad s > 0.$$

<u>Ex (V)</u>: Laplace transform of a discontinuous function.

$$f(t) = \begin{cases} 1, & 0 \le t < 1 \\ k & t = 1 \\ 0 & t > 1 \end{cases}, \qquad K = \text{constant}.$$

$$\mathcal{L}\{f(t)\} = \int_0^\infty e^{-st} f(t)\, dt$$



$$= \int_0^1 e^{-st} \cdot 1\, dt + \int_1^\infty e^{-st} \cdot 0 \cdot dt$$

$$= \int_0^1 e^{-st} dt = -\frac{e^{-st}}{s}\Big|_0^1 = \frac{1-e^{-s}}{s}, \quad s > 0.$$

Observe that $\mathcal{L}\{f(t)\}$ does not depend on the exact value at $t=1$, ''; K. Thus there could be two functions differing in their value at a single point, but have the same Laplace transform.

Ex: (vi) Find Laplace transform of $f(t) = 5e^{-2t} - 3\sin(4t)$, $t \geq 0$

Since $\mathcal{L}\{c_1 f_1(t) + c_2 f_2(t)\} = c_1 \mathcal{L}\{f_1(t)\} + c_2 \mathcal{L}_2\{f_2(t)\}$, we have

$$\mathcal{L}\{5e^{-2t} - 3\sin 4t\} = 5\mathcal{L}\{e^{-2t}\} - 3\mathcal{L}\{\sin 4t\}$$

Since $\mathcal{L}\{e^{at}\} = \dfrac{1}{s-a}$, $s > a$, we have

$$\mathcal{L}\{e^{-2t}\} = \frac{1}{s+2}, \quad s > -2.$$

Also $\mathcal{L}\{\sin at\} = \dfrac{a}{s^2+a^2}$, $s > 0$

$$\Rightarrow \mathcal{L}\{\sin 4t\} = \frac{4}{s^2+16}, \quad s > 0$$

Combing, we have

$$F(s) = \frac{5}{s+2} - \frac{12}{s^2+16}, \quad \underline{\underline{s > 0}}.$$

§ 6.2 Solution of Initial Value Problems

Before we begin to use Laplace transforms to solve differential equations, we first need to evaluate Laplace transforms of derivatives.

If $f(t)$ has a Laplace transform $F(s)$, what is the Laplace transform of $\frac{df}{dt}$?

Theorem 6.2.1 If f is continuous and of exponential order, and if f' is ~~contin~~ piecewise continuous on $[0,\infty)$, then

$$\mathcal{L}\{f'(t)\} = s\,\mathcal{L}\{f(t)\} - f(0)$$

Proof: $$\mathcal{L}\{f'(t)\} = \int_0^\infty e^{-st} f'(t)\,dt = \lim_{R\to\infty} \int_0^R e^{-st} f'(t)\,dt$$

Using integration by parts, we have

$$\mathcal{L}\{f'(t)\} = e^{-st} f(t)\Big|_0^R - \int_0^R -s\,e^{-st} f(t)\,dt$$

$$= \left\{e^{-sR} f(R) - f(0)\right\} + s\int_0^R e^{-st} f(t)\,dt$$

Taking the limit $R\to\infty$, we have

$$\mathcal{L}\{f'(t)\} = \lim_{R\to\infty} e^{-sR} f(R) - f(0) + s\,\mathcal{L}\{f(t)\}$$

Since $f(t)$ is of exponential order, ~~exax~~ $s > a$

ie; $f(t) \le Ke^{at}$, $e^{-sR} f(t) = e^{-(s-a)R} \cdot K$

$\therefore \lim_{R \to \infty} e^{-sR} f(R) = 0$ for $s > a$.

$$\therefore \boxed{\mathcal{L}\{f'(t)\} = s\mathcal{L}\{f(t)\} - f(0)}$$

If $y(t) = f(t)$, we have

$$\mathcal{L}\{y'\} = s\mathcal{L}\{y\} - y(0)$$

$$= sY(s) - y(0)$$

where $Y(s) = \mathcal{L}\{y(t)\}$ is the Laplace transform of $y(t)$.

$\mathcal{L}\{y''\} = ?$ $\mathcal{L}\{y''\} = \mathcal{L}\{(y')'\}$

Let $y' = f(t)$, then

$$\mathcal{L}\{y''\} = \mathcal{L}\{f'(t)\} = s\mathcal{L}\{f(t)\} - f(0)$$

$$= s\mathcal{L}\{y'\} - y'(0)$$

$$= s\left[s\mathcal{L}\{y\} - y(0)\right] - y'(0)$$

$$= s^2\mathcal{L}\{y\} - sy(0) - y'(0)$$

$$\therefore \boxed{\mathcal{L}\{y''\} = s^2 Y(s) - sy(0) - y'(0)}$$

Similarly $\mathcal{L}\{y^{(n)}\} = s^n Y(s) - s^{n-1}y(0) \dots - sy^{(n-2)}(0) - y^{(n-1)}(0)$

Solving a differential equation : General outline :

Consider the IVP

$$y'' + py' + qy = g(t)$$

$$y'(0) = y_0 \; ; \; y'(0) = y_0'$$

Step 1: Take the Laplace transform of the equation :

$$\mathcal{L}\{y'' + py' + qy\} = \mathcal{L}\{g\}$$

Using the formulae for Laplace transform of y' & y''.

Step 2: Get an equation for $Y(s) = \mathcal{L}\{y(t)\}$.

Typically, we get $Y(s) = \dfrac{A(s)}{B(s)}$.

We can use ~~Laplace transform~~ partial fractions to reduce $Y(s)$ into simple quantities.

Step 3: Obtain $y(t)$ as the inverse Laplace transform of $Y(s)$, ie; $y(t) = \mathcal{L}^{-1}\{Y(s)\}$.

In obtaining $Y(s)$, the initial conditions are used $\Longrightarrow$ initial conditions are automatically satisfied.

Let us see the above procedure with an example.

Ex:
$$y'' - y' - 2y = 0$$
$$y(0) = 1 \quad ; \quad y'(0) = 0$$

Step 1:
$$\mathcal{L}\{y'' - y' - 2y\} = \mathcal{L}\{0\}$$

Since $\mathcal{L}\{f(t)\} = \int_0^\infty e^{-st} f(t)\, dt$, $\mathcal{L}\{0\} = 0$.

Recall $\mathcal{L}\{y'\} = sY(s) - y(0)$ where $Y(s)$ is the Laplace transform of $y(t)$.

$\mathcal{L}\{y''\} = s^2 Y(s) - sy(0) - y'(0)$.

Here $y(0)$ & $y'(0)$ are the initial conditions. We get

Step 2:
$$[s^2 Y(s) - s\,y(0) - y'(0)] - [s\,Y(s) - y(0)] - 2\,Y(s) = 0$$

$$\Rightarrow (s^2 - s - 2)\,Y(s) + (1-s)\underbrace{y(0)}_{=1} - \underbrace{y'(0)}_{=0} = 0$$

$$\Rightarrow \underbrace{(s^2 - s - 2)}\,Y(s) + (1-s) = 0$$

: Initial condition already applied.

Notice that this is like the characteristic equation you saw before.

$$\Rightarrow Y(s) = \frac{s-1}{s^2-s-2} = \frac{s-1}{(s-2)(s+1)}$$

Using partial fractions, we can write

$$Y(s) = \frac{A}{s-2} + \frac{B}{s+1}.$$

We get $A = \frac{1}{3}$, $B = \frac{2}{3}$.

$$\Rightarrow \boxed{Y(s) = \frac{(1/3)}{s-2} + \frac{(2/3)}{s+1}}$$

<u>Step 3</u>: Use $Y(s)$ to obtain $y(t)$ using an inverse Laplace Transform, i.e.;

$$y(t) = \mathcal{L}^{-1}\{Y(s)\} = \mathcal{L}^{-1}\left\{\frac{1}{3}\cdot\frac{1}{(s-2)} + \frac{2}{3}\frac{1}{(s+1)}\right\}$$

$$= \frac{1}{3}\mathcal{L}^{-1}\left\{\frac{1}{s-2}\right\} + \frac{2}{3}\mathcal{L}^{-1}\left\{\frac{1}{s+1}\right\}$$

Since $\mathcal{L}\{e^{at}\} = \frac{1}{s-a}$, $\mathcal{L}^{-1}\left\{\frac{1}{s-a}\right\} = e^{at}$

$\therefore \mathcal{L}^{-1}\left\{\frac{1}{s-2}\right\} = e^{2t}$ and $\mathcal{L}^{-1}\left\{\frac{1}{s+1}\right\} = e^{-t}$

$$\therefore y(t) = \frac{1}{3}\cdot e^{2t} + \frac{2}{3}e^{-t}$$

<u>NOTE</u>: Notice that $Y(s)$ has "poles" or singularities at $s = -1$ and $s = 2$. These singularities occur exactly at the roots of the characteristic equation.

<u>Verify</u> the <u>above</u> <u>result</u> using <u>a characteristic equation</u>:-

$y'' - y' - 2y = 0$

Let $y = e^{rt}$ $\Rightarrow$ $r^2 - r - 2 = 0$ $\Rightarrow$ $(r-2)(r+1) = 0$

$\Rightarrow r_1 = 2, \quad r_2 = -1$

$\therefore \quad y_1 = e^{2t} \quad ; \quad y_2 = e^{-t}$

$\therefore \quad y(t) = c_1 e^{2t} + c_2 e^{-t}.$

Now $\quad y(0) = 1 \Rightarrow 1 = c_1 + c_2$
& $\quad y'(0) = 0 \Rightarrow 0 = 2c_1 - c_2$ $\Big\} \; c_1 = \frac{1}{3}, \quad c_2 = \frac{2}{3}$

$\therefore \quad y(t) = \frac{1}{3} e^{2t} + \frac{2}{3} e^{-t}$

Ex: $\quad y'' + y = \sin(2t)$

$y(0) = 2 \; ; \quad y'(0) = 1$

Step 1. $\quad \mathcal{L}\{ y'' + y \} = \mathcal{L}\{ \sin 2t \}$

Recall that $\mathcal{L}\{\sin at\} = \frac{a}{s^2+a^2} \Rightarrow \mathcal{L}\{\sin 2t\} = \frac{2}{s^2+4}$

$\Rightarrow \quad [s^2 Y(s) - s\, y(0) - y'(0)] + Y(s) = \frac{2}{s^2+4}$

$\Rightarrow \quad s^2 Y(s) - 2s - 1 + Y(s) = \frac{2}{s^2+4}$

Step 2: $\quad \Rightarrow (s^2+1)\, Y(s) = \frac{2}{s^2+4} + 1 + 2s = \frac{2s^3 + s^2 + 8s + 6}{s^2+4}$

$\therefore \quad Y(s) = \frac{2s^3 + s^2 + 8s + 6}{(s^2+1)(s^2+4)} \quad \text{——— } (*)$

Using partial fractions, $\quad Y(s) = \frac{as+b}{s^2+1} + \frac{cs+d}{s^2+4}$

$= \frac{(as+b)(s^2+4) + (cs+d)(s^2+1)}{(s^2+1)(s^2+4)}$

$\text{——— } (**)$

Comparing similar terms in (*) and (**), we have

$a = 2, \quad b = \frac{5}{3}, \quad c = 0, \quad d = \frac{-2}{3}.$

$$\therefore \quad Y(s) = \frac{2s + \frac{5}{3}}{s^2+1} + \frac{(-2/3)}{s^2+4}$$

$$= 2 \cdot \frac{s}{s^2+1} + \frac{5}{3} \cdot \frac{1}{s^2+1} - \frac{2}{3} \cdot \frac{1}{s^2+4}$$

Step 3:

$\therefore \quad y(t) = \mathcal{L}^{-1}\{Y(s)\}.$

Since $\mathcal{L}\{\cos at\} = \frac{s}{s^2+a^2}$; $\mathcal{L}^{-1}\left\{\frac{s}{s^2+a^2}\right\} = \cos at$

Similarly $\mathcal{L}^{-1}\left\{\frac{a}{s^2+a^2}\right\} = \sin at$

$$\therefore \quad y(t) = 2\,\mathcal{L}^{-1}\underbrace{\left\{\frac{s}{s^2+1}\right\}}_{a=1} + \frac{5}{3}\,\mathcal{L}^{-1}\underbrace{\left\{\frac{1}{s^2+1}\right\}}_{a=1} - \frac{1}{3}\,\mathcal{L}^{-1}\underbrace{\left\{\frac{2}{s^2+4}\right\}}_{a=2}$$

$$= 2\cos t + \frac{5}{3}\sin t - \frac{1}{3}\sin(2t)$$

Ex: Higher Order equation :-

$$y'''' - y = 0$$

$y(0) = 0$; $y'(0) = 1$; $y''(0) = 0$; $y'''(0) = 0$

step 1: $\mathcal{L}\{y''''\} = s^4 Y(s) - s^3 y(0) - s^2 y'(0) - s y''(0) - y'''(0)$

$= s^4 Y(s) - 0 \quad - s^2 \quad - 0$

Step 2:

$$\Rightarrow \quad s^4 Y(s) - s^2 - Y(s) = 0$$

$$\Rightarrow \quad Y(s)\left[s^4 - 1\right] = s^2$$

$$\Rightarrow \quad Y(s) = \frac{s^2}{s^4 - 1} = \frac{s^2}{(s^2-1)(s^2+1)}$$

Using partial fractions, we have

$$Y(s) = \frac{as+b}{s^2-1} + \frac{cs+d}{s^2+1}.$$

with $a = 0,\ b = \frac{1}{2},\ c = 0,\ d = \frac{1}{2}$, we have

$$Y(s) = \frac{1/2}{s^2-1} + \frac{1/2}{s^2+1}$$

Step 3: Using the table of Laplace transforms, we find that

$$\mathcal{L}\{\sinh at\} = \frac{a}{s^2-a^2} \Rightarrow \mathcal{L}^{-1}\left\{\frac{1}{s^2-1}\right\} = \sinh t$$

$$\mathcal{L}^{-1}\left\{\frac{1}{s^2+1}\right\} = \sin t$$

$$\therefore\ y(t) = \mathcal{L}^{-1}\{Y(s)\}$$

$$= \frac{1}{2}\sinh(t) + \frac{1}{2}\sin(t)$$

$\sinh(t)$ and $\cosh(t)$ are hyperbolic sin and cosine functions. They are related to exponentials in the following way:

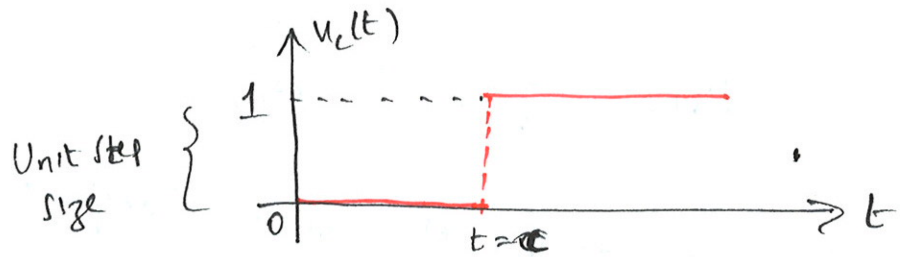
$$\sinh t = \frac{e^t - e^{-t}}{2}$$

$$\cosh t = \frac{e^t + e^{-t}}{2}$$

§ 6.3 STEP FUNCTIONS

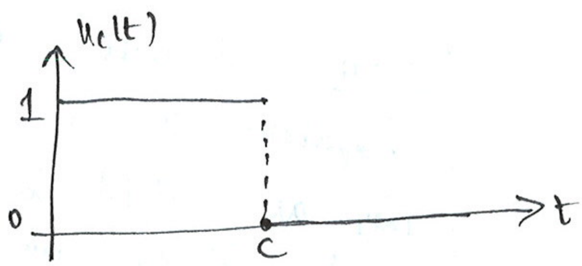
It is common to encounter differential equations with discontinuous forcing. In order to deal easily with such discontinuous functions (ie; functions with jumps), we define a new function called the unit step function or Heaviside function.

$$u_c(t) = \begin{cases} 0 & \text{for } t < c \\ 1 & \text{for } t \geq c, \end{cases} \quad \text{where } c \geq 0.$$



Using $u_c(t)$, we can define other step functions:

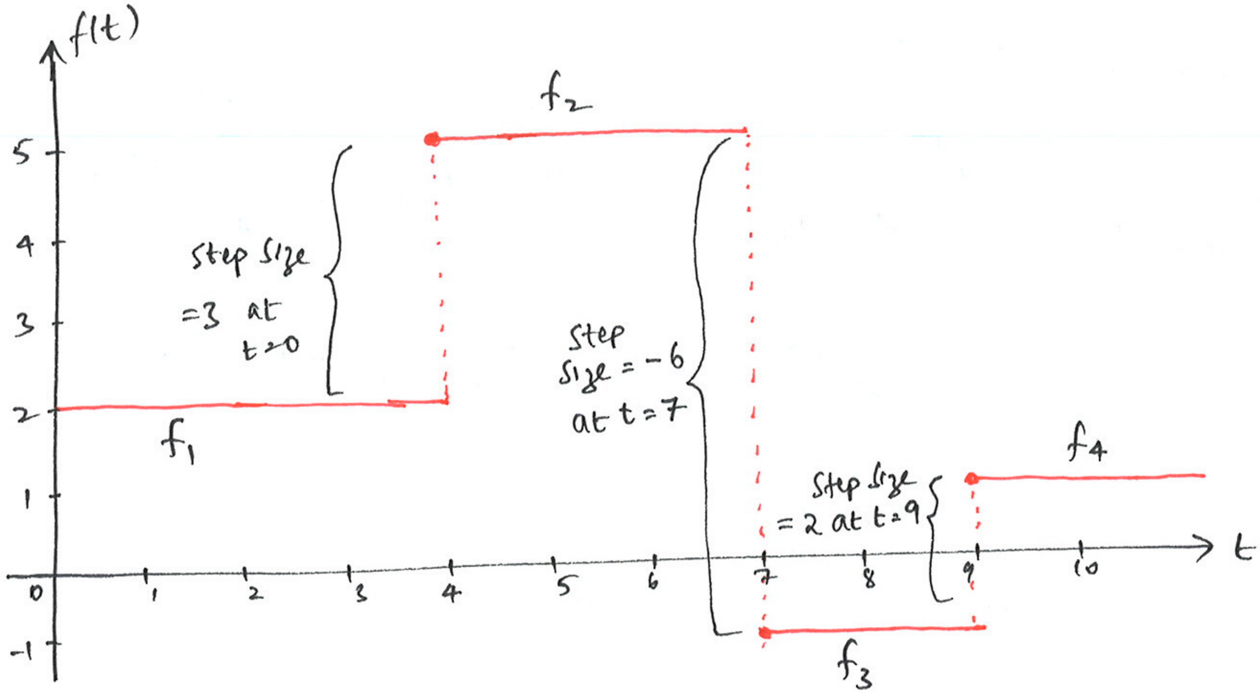
Ex:



$\longrightarrow y = 1 - u_c(t)$

Ex: Sketch $f(t)$ and express in terms of $u(t)$.

$$f(t) = \begin{cases} 2 & 0 \leq t < 4 \\ 5 & 4 \leq t < 7 \\ -1 & 7 \leq t < 9 \\ 1 & t \geq 9 \end{cases}$$



$f_1 = 2$

From f_1 to f_2, there is a jump of 3 at $t = 4$

$$\Rightarrow \quad f_2 = f_1 + 3u_4(t) = 2 + 3u_4(t)$$

$$\left[u_4(t) = \begin{cases} 0 & \text{for } t < 4 \\ 1 & \text{for } t > 4 \end{cases} \right]$$

From f_2 to f_3, there is a jump of -6 at $t = 7$.

$$\Rightarrow \quad f_3 = f_2 - 6u_7(t) = 2 + 3u_4(t) - 6u_7(t)$$

From f_3 to f_4, there is a jump of $+2$ at $t = 9$

$$\Rightarrow \quad f_4 = f_3 + 2u_9(t) = 2 + 3u_4(t) - 6u_7(t) + 2u_9(t).$$

The final function $f_4(t)$ is also $f(t)$.

$$\therefore \quad f(t) = 2 + 3u_4(t) - 6u_7(t) + 2u_9(t).$$

Laplace transform of $u_c(t) = ?$

$$\mathcal{L}\{u_c(t)\} = \int_0^\infty e^{-st} u_c(t)\, dt = \int_0^c e^{-st} \cdot 0 + \int_c^\infty e^{-st} \cdot 1 \, dt$$

$$= \left. \frac{e^{-st}}{-s} \right|_c^\infty = \frac{-1}{s}\left[0 - e^{-cs}\right], \quad s > 0$$

$$= \frac{e^{-cs}}{s}, \quad s > 0$$

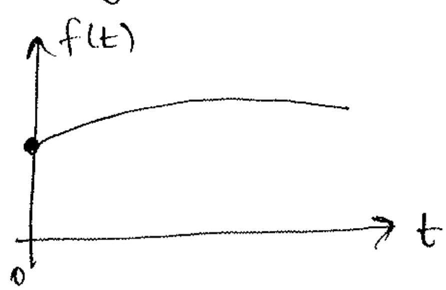
Ex: Calculate the Laplace transform of $f(t)$ from the previous example:

$$f(t) = 2 + 3u_4(t) - 6u_7(t) + 2u_9(t)$$

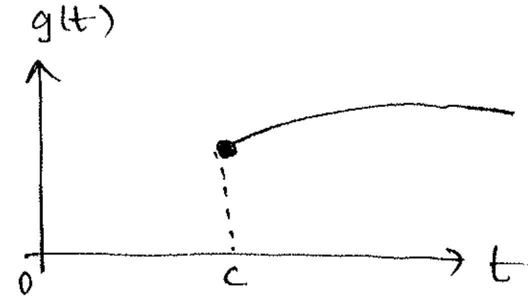
$$F(s) = \mathcal{L}\{f(t)\} = \mathcal{L}\{2\} + 3\mathcal{L}\{u_4(t)\} - 6\mathcal{L}\{u_7(t)\} + 2\mathcal{L}\{u_9(t)\}$$

$$= \frac{2}{s} + 3\frac{e^{-4s}}{s} - 6\frac{e^{-7s}}{s} + \frac{2e^{-9s}}{s}$$

Translating a function:-



$\longrightarrow$



We have shifted $f(t)$ by a distance 'c' to obtain $g(t)$

$$\Rightarrow g(t) = \begin{cases} 0 & \text{for } t < c \\ f(t-c) & \text{for } t \geq c \end{cases}$$

In terms of the unit step function, we have

$$g(t) = u_c(t)\, f(t-c).$$

<u>Theorem 6.3.1</u>: If $F(s) = \mathcal{L}\{f(t)\}$, and if c is a positive constant, then $\mathcal{L}\{u_c(t)\, f(t-c)\} = e^{-cs}\mathcal{L}\{f(t)\} = e^{-cs}F(s)$.

Conversely, $u_c(t)\, f(t-c) = \mathcal{L}^{-1}\{e^{-cs}F(s)\}$

<u>Proof</u>: $$\mathcal{L}\{u_c(t)\, f(t-c)\} = \int_0^\infty u_c(t)\, f(t-c) \cdot e^{-st}\, dt$$

Since $u_c(t) = 0$ for $t < c$, we have

$$\mathcal{L}\{u_c(t)\, f(t-c)\} = \int_c^\infty f(t-c)\, e^{-st}\, dt.$$

Let $\xi = t - c \Rightarrow dt = d\xi$

$$\Rightarrow \int_{\xi=0}^{\xi=\infty} f(\xi)\, e^{-s(\xi+c)}\, d\xi = e^{-cs}\underbrace{\int_0^\infty e^{-s\xi} f(\xi)\, d\xi}_{\text{same as } F(s)}$$

$$\Rightarrow \mathcal{L}\{u_c(t)\, f(t-c)\} = e^{-cs}F(s)$$

Ex: If $f(t) = 1$,

$$\mathcal{L}\{u_c(t) f(t-c)\} = \mathcal{L}\{u_c(t)\cdot 1\}$$
$$= \mathcal{L}\{u_c(t)\}$$

Using theorem 6.3.1, $\mathcal{L}\{u_c(t) f(t-c)\} = e^{-cs} F(s)$

Since $F(s) = \mathcal{L}\{f(t)\} = \frac{1}{s}$, we have

$$\mathcal{L}\{u_c(t)\} = \frac{e^{-cs}}{s}.$$

Ex: Find the Laplace transform of f(t) where

$$y(t) = \begin{cases} \sin t, & 0 \le t < \pi/4 \\ \sin t + \cos(t - \frac{\pi}{4}), & t \geq \pi/4 \end{cases}$$

We can rewrite f(t) as

$y(t) = \sin t + g(t)$ where $g(t) = \begin{cases} 0 & 0 \le t < \pi/4 \\ \cos(t - \frac{\pi}{4}) & t \geq \pi/4 \end{cases}$

$$= u_{\frac{\pi}{4}}(t)\cos(t - \frac{\pi}{4})$$

$$\mathcal{L}\{y(t)\} = \mathcal{L}\{\sin t\} + \mathcal{L}\{g(t)\}$$
$$= \underbrace{\frac{1}{s^2+1}}_{\text{(from \#5 of the table)}} + \mathcal{L}\{g(t)\}$$

$$\mathcal{L}\{g(t)\} = \mathcal{L}\left\{u_{\frac{\pi}{4}}(t)\cos\left(t-\frac{\pi}{4}\right)\right\}$$

$$= \mathcal{L}\{u_c(t)\, f(t-c)\} \quad \text{where} \quad c = \frac{\pi}{4}$$

$$f(t) = \cos t$$

$$= e^{-cs} F(s) \quad \text{where} \quad F(s) = \mathcal{L}\{f(t)\}$$

$$= \mathcal{L}\{\cos t\} = \frac{s}{s^2+1}$$

$$= e^{-\frac{\pi}{4}s} \cdot \frac{s}{s^2+1}$$

$$\therefore \mathcal{L}\{y(t)\} = \frac{1}{s^2+1} + \frac{s e^{-\frac{\pi}{4}s}}{s^2+1}$$

Ex: Find the inverse Laplace transform of

$$F(s) = \frac{1-e^{-2s}}{s^2}$$

$$f(t) = \mathcal{L}^{-1}\{F(s)\} = \mathcal{L}^{-1}\left\{\frac{1-e^{-2s}}{s^2}\right\}$$

$$= \mathcal{L}^{-1}\left\{\frac{1}{s^2}\right\} - \mathcal{L}^{-1}\left\{\frac{e^{-2s}}{s^2}\right\}$$

From the table, #3, $\mathcal{L}\{t^n\} = \frac{n!}{s^{n+1}}$. Therefore, if $n = 1$,

$$\mathcal{L}\{t\} = \frac{1}{s^2} \Rightarrow \mathcal{L}^{-1}\left\{\frac{1}{s^2}\right\} = t$$

What is $\mathcal{L}^{-1}\left\{\frac{e^{-2s}}{s^2}\right\}$. This is of the form

$$\mathcal{L}^{-1}\{e^{-cs} F(s)\} \quad \text{where} \quad F(s) = \frac{1}{s^2}, \quad c = 2$$

From the table, #13, $\mathcal{L}^{-1}\{e^{-cs}F(s)\} = u_c(t)\, f(t-c)$

where $f(t) = \mathcal{L}^{-1}\{F(s)\}$

(different from the previous $f(t)$) $= \mathcal{L}^{-1}\left\{\frac{1}{s^2}\right\} = t \Rightarrow f(t-c) = t-c$

$\therefore \mathcal{L}^{-1}\left\{\frac{e^{-2s}}{s^2}\right\} = u_2(t)\,(t-2)$

$\therefore \mathcal{L}^{-1}\left\{\frac{1-e^{-2s}}{s^2}\right\} = t - u_2(t)(t-2)$

Ex: Find inverse transform of

$$G(s) = \frac{1}{s^2-4s+5}$$

$g(t) = \mathcal{L}^{-1}\left\{\frac{1}{s^2-4s+5}\right\} = \mathcal{L}^{-1}\left\{\frac{1}{(s-2)^2+1}\right\} = \mathcal{L}^{-1}\{F(s-2)\}$

where $F(s) = \frac{1}{s^2+1}$

From #14, $\mathcal{L}^{-1}\{F(s-2)\} = e^{2t}f(t)$ where $f(t) = \mathcal{L}^{-1}\{F(s)\}$

$= e^{2t}.\sin t$

$\therefore g(t) = e^{2t}\sin t$

Problem #22:

$F(s) = \frac{2e^{-2s}}{s^2-4} = 2e^{-2s}G(s)$ where $G(s) = \frac{1}{s^2-4}$

From #13 of the table, $\mathcal{L}^{-1}\{e^{-cs}G(s)\} = u_c(t)\, g(t-c)$

where $g(t) = \mathcal{L}\{G(s)\}$

Here $c=2$, $g(t) = \mathcal{L}\left\{\frac{1}{s^2-4}\right\} = \frac{1}{2}\sinh(2t) \Rightarrow f(t) = u_2(t)\sinh[2(t-2)]$

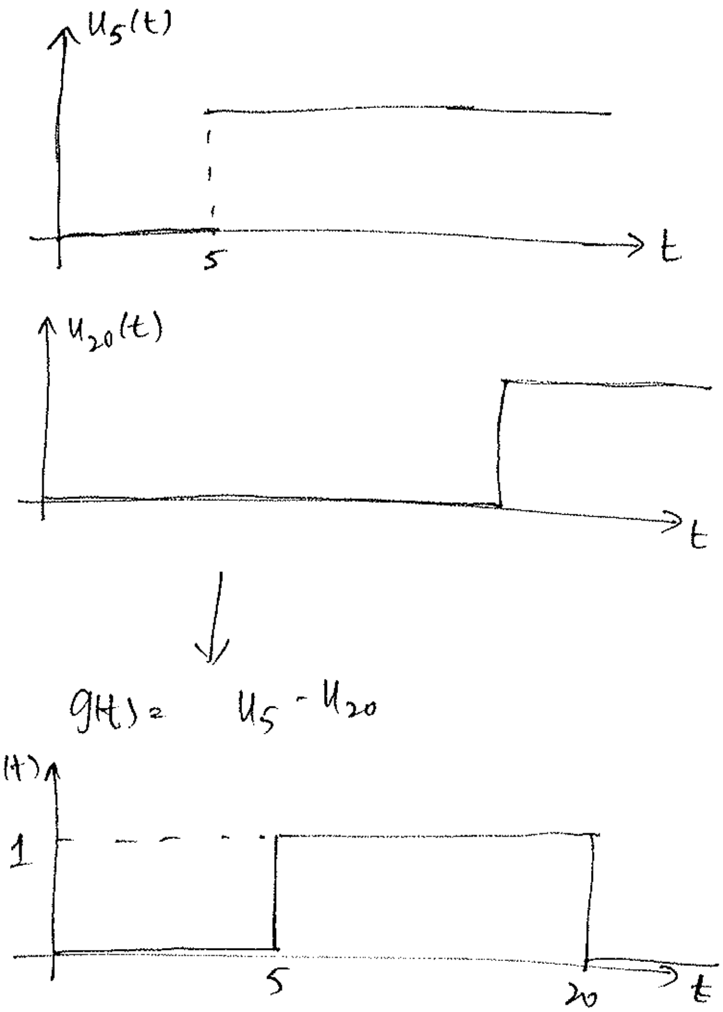
§ 6.4 Solving Equations with Discontinuous Forcing

Ex: $2y'' + y' + 2y = g(t)$; $y(0) = 0$; $y'(0) = 0$

where $g(t) = u_5(t) - u_{20}(t)$

Take the Laplace transform of the equation:

$$2\mathcal{L}\{y''\} + \mathcal{L}\{y'\} + 2\mathcal{L}\{y\} = \mathcal{L}\{g(t)\}$$



Recall,

$$\mathcal{L}\{y''\} = s^2 Y(s) - s\,y(0) - y'(0)$$

$$\mathcal{L}\{y'\} = s\,Y(s) - y(0)$$

where $Y(s) = \mathcal{L}\{y\}$.

We therefore get

$$2\left[s^2 Y(s) - s \cdot 0 - 0\right] + \left[sY(s) - 0\right] + 2Y(s) = \mathcal{L}\{g(t)\}$$

$$\Rightarrow (2s^2 + s + 2)\,Y(s) = \mathcal{L}\{g(t)\}$$

$$\mathcal{L}\{g(t)\} = \mathcal{L}\{u_5(t)\} - \mathcal{L}\{u_{20}(t)\} = \frac{e^{-5s}}{s} - \frac{e^{-20s}}{s}$$

$$\Rightarrow Y(s) = \frac{e^{-5s} - e^{-20s}}{s(2s^2 + s + 2)} = \left(e^{-5s} - e^{-20s}\right) H(s)$$

where $H(s) = \dfrac{1}{s(2s^2 + s + 2)}$

If $h(t) = \mathcal{L}^{-1}\{H(s)\}$, we have

$$y(t) = \mathcal{L}^{-1}\{Y(s)\} = \mathcal{L}^{-1}\{e^{-5s}H(s)\} - \mathcal{L}^{-1}\{e^{-20s}H(s)\}$$

$$= u_5(t)\, h(t-5) - u_{20}(t)\, h(t-20)$$

What is $h(t)$? $\quad h(t) = \mathcal{L}^{-1}\left\{\dfrac{1}{s(2s^2+s+2)}\right\}$

$$H(s) = \frac{a}{s} + \frac{bs+c}{2s^2+s+2}$$

We get $a = \frac{1}{2}$, $b = -1$, $c = -\frac{1}{2}$

$$\Rightarrow H(s) = \frac{1}{2}\cdot\left(\frac{1}{s}\right) - \frac{s+\frac{1}{2}}{2s^2+s+2}$$

$$= \frac{1}{2}\cdot\frac{1}{s} - \frac{s+\frac{1}{2}}{2\left[\left(s+\frac{1}{4}\right)^2 + \frac{15}{16}\right]}$$

$$= \frac{1}{2}\cdot\left(\frac{1}{s}\right) - \frac{1}{2}\cdot\frac{\left(s+\frac{1}{4}\right)+\frac{1}{4}}{\left(s+\frac{1}{4}\right)^2 + \left(\frac{\sqrt{15}}{4}\right)^2}$$

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$2s^2+s+2 = 2\left(s^2+\frac{1}{2}s+1\right)$

$= 2\left[\left(s+\frac{1}{4}\right)^2 + \frac{15}{16}\right]$

$s+\frac{1}{2} = s+\frac{1}{4}+\frac{1}{4}$

$\mathcal{L}\{1\} = \frac{1}{s} \Rightarrow \mathcal{L}^{-1}\left\{\frac{1}{s}\right\} = 1$

$$\therefore h(t) = \frac{1}{2}\cdot 1 - \frac{1}{2}\,\mathcal{L}^{-1}\left\{\frac{s+\frac{1}{4}}{\left(s+\frac{1}{4}\right)^2+\left(\frac{\sqrt{15}}{4}\right)^2}\right\}$$

$$- \frac{1}{2}\,\frac{1}{4}\cdot\mathcal{L}^{-1}\left\{\frac{\frac{\sqrt{15}}{4}}{\left(s+\frac{1}{2}\right)^2+\left(\frac{\sqrt{15}}{4}\right)^2}\right\}\frac{1}{\left(\frac{\sqrt{15}}{4}\right)}$$

$$= \frac{1}{2} - \frac{1}{2}e^{-t/4}\cos\left(\frac{\sqrt{15}}{4}t\right) - \frac{1}{2\sqrt{15}}e^{-t/4}\sin\left(\frac{\sqrt{15}}{4}t\right)$$