MATH 215/255: Elementary Differential Equations I
Homework - 1, Due: 14-Sep-2012

First name: **Last name:**

Student number:

Note: *Write your answers clearly in the space provided.*

I. Classification

Ordinary Differential Equations (in short, ODE) are equations in which the unknown function (dependent variable), typically, $y(x)$ or $y(t)$, depends on just one independent variable, x or t. These are the kind that were discussed in class. For the following ODE's, determine the order of the equation and state whether the equation is linear or nonlinear. Write your answers in the space below each equation.

Problems 1-5:

1. $\dfrac{dy}{dx} = \dfrac{x}{y}$,

   Order:
   Linear/Nonlinear:

2. $x^2 \dfrac{d^2y}{dx^2} + 4\dfrac{dy}{dx} + y = 10x$,

   Order:
   Linear/Nonlinear:

3. $a(x)\dfrac{d^2y}{dx^2} + b(x)\dfrac{dy}{dx} + c(x)y = d(x)$,

   Order:
   Linear/Nonlinear:

4. $M(x,y)\dfrac{dy}{dx} + N(x,y) = 0$,

   Order:
   Linear/Nonlinear:

5. $x^2\frac{d^2y}{dx^2} + x\frac{dy}{dx} + (x^2 - \alpha^2)y = 0, \quad \alpha = \text{constant}$: This is the famous Bessel's equation which appears in many branches of physics.

   Order:
   Linear/Nonlinear:

Problem 6:

Let's now look at a slightly more complicated example. Newton's second law of motion relates the acceleration of particle of mass m to the external force, $\mathbf{F}$ acting on it. Often, the force on the particle depends on the position of the particle. In this case, Newton's second law can be written as

$$m\frac{d^2\mathbf{r}(t)}{dt^2} = \mathbf{F}(\mathbf{r}(t), t),$$

where $\mathbf{r}(t)$ is the vector position of the particle. This is clearly a second order equation. Determine the conditions in which this equation is linear and nonlinear. Write your answers in the space below.

Condition for linear equation:

Condition for nonlinear equation:

The other class of differential equations are equations where the unknown function can depend on many independent variables. Instead of regular derivatives, we now have to use partial derivatives, hence these equations are called Partial Differential Equations (PDE's). Many real life problems are governed by PDE's. In this course, we'll not be studying them. For the equations below, again determine the order and linearity of the equations.

Problem 7-9:

7. $\frac{\partial T}{\partial t} = \alpha\frac{\partial^2 T}{\partial x^2}$: This is the Heat equation which describes the distribution of temperature T as a function of time, t, and space, x, and α is the thermal diffusivity (which is a constant here).

   Order:
   Linear/Nonlinear:

8. $\frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} = 0$: This is called the Laplace equation.

Order:
Linear/Nonlinear:

9. $\dfrac{\partial u}{\partial t} + u\dfrac{\partial u}{\partial x} = \nu\dfrac{\partial^2 u}{\partial x^2}$: This is the Burger's equation which is used to model Shock Waves.

Order:
Linear/Nonlinear:

II. Radioactive decay

Consider a sample of some radioactive material (say Uranium-234) with N_0 being the initial mass of the sample. As radioactive radiation is emitted, the sample decays at a rate proportional to the amount currently present. If $N(t)$ is the size of the sample at any time t, then the above statement can be written mathematically as

$$\frac{d}{dt}N(t) \propto -N(t),$$

where the negative sign indicates that this is a decay process. If λ is the proportionality constant, called the decay constant, this equation becomes

$$\frac{d}{dt}N(t) = -\lambda N(t),$$

10 (i). Solve the above equation and determine the integration constant using the condition $N(t=0) = N_0$.
10 (ii). The time required for the sample size to reduce from N_0 to half its size, i.e., $N_0/2$ is called the half-life of the material, $t_{1/2}$. Obtain an expression for $t_{1/2}$ in terms of the decay rate, λ.
10 (iii). If the initial sample is $N_0 = 1$ gram and it decays to 0.6 grams in 100 seconds, determine the decay rate λ.

III. Population growth

Now let's look at a problem which is inverse of the radioactivity problem. Consider a population, $p(t)$, of some lucky rabbits in a field without any predators. If the rate of growth of the population is proportional to the current population, i.e. $dp/dt = rp$, where r is the growth rate,

11(i). Determine the rate constant r if the population doubles in 30 days.

11(ii). Luck changes to misfortune. Now, the rabbits have contracted a disease and die at a constant rate of k rabbits per day. Now the equation becomes $dp/dt = rp - k$. If the initial population is $p(0) = 50$, growth rate $r = 0.5$ and death rate is $k = 30$, how long will it take for the rabbits to become extinct? (Note that r is now different and is unrelated to part (i)).

III. Direction fields

12. Draw the direction field for the following problem and determine the behaviour of y as $t \to \infty$. (Note: y' is same as $\dfrac{dy}{dx}$).

$$y' = 3 - 2y.$$

13. Draw the direction field for the following problem. In this case, the solution as $t \to \infty$ will depend on the where you start at $t = 0$. Explain this dependence in words.

$$y' = y(y - 3).$$