# Existence and Uniqueness Theorem (For first order equations)

Differential equations are used to model many physical phenomenon. Infact, many equations are named after the phenomenon for which the differential equation is used. Even if we ~~not~~ may not be in a position to solve the equation, we can still find out about the "nature" of solutions. This is where Existence and Uniqueness theorem comes in.

**Existence:** Does the given differential equation have a solution or not?

**Uniqueness:** For a given initial value problem (equation with a given initial condition), is there a unique solution or are there many solutions?

**Existence and Uniqueness theorem for first order linear differential equation:-**

Consider the differential equation

$$\left.\begin{aligned} & y' + p(t)\,y = g(t) \\ & \text{with } y(t_0) = y_0 \end{aligned}\right\} \text{IVP}$$

If $p(t)$ and $q(t)$ are continuous on an interval $t \in (\alpha, \beta)$ such that $t_0 \in (\alpha, \beta)$, then there exists a unique solution for any initial condition $y(t_0) = y_0$, and the solution exists throughout the interval $(\alpha, \beta)$.

Proof trivially follows using the method of integrating factors.

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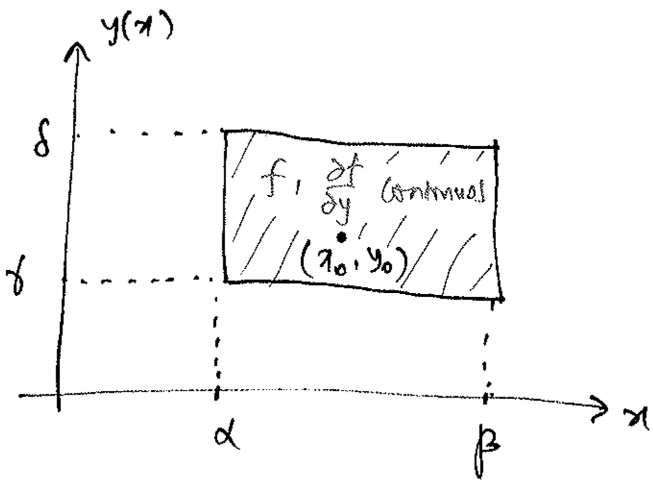
Existence and Uniqueness theorem for first order nonlinear differential equation

Consider a nonlinear differential equation

$$\text{IVP} \begin{cases} y' = f(x, y) \\ y(x_0) = y_0 \end{cases}$$




Suppose $f(x,y)$ and $\frac{\partial f}{\partial y}(x,y)$ are continuous on some rectangular region $x \in (\alpha, \beta)$, $y(x) \in (\gamma, \delta)$, and the initial condition $(x_0, y_0)$ also be inside the rectangle, then the given IVP has a existence unique solution.

Proof is much harder for this case.

What happens if these conditions are not obeyed? ($f$ or $\frac{\partial f}{\partial y}$ may not be continuous)

If the above conditions are not met, then either

→ we ~~may not~~ may not have a solution to the given IVP

→ or, the solution may not be unique, i.e; there could be more than one solution with the same initial conditions.

Lets see examples where such failures happen.

Ex 1
$$\left.\begin{aligned} \frac{dy}{dx} &= y^{1/3} \\ y(0) &= 0 \,; \quad x \geq 0 \end{aligned}\right\} \text{IVP}$$
: First order nonlinear diff. equation.

This is in the form

$$y' = f(x,y)$$

where $f(x,y) = y^{1/3}$ ; $\frac{\partial f}{\partial y} = \frac{1}{3} y^{-2/3}$

(i) $f(x,y)$ is continuous for all $x \geq 0$

(ii) $\frac{\partial f}{\partial y}$ does $\underbrace{\text{not exist}}_{\text{(not continuous at } y=0)}$ when $y = 0$.

Therefore, the Existence and Uniqueness theorem does not apply to this problem. Let us solve the problem and see how this failure manifests itself.

$$\frac{dy}{dx} = y^{1/3}$$

$$\Rightarrow \frac{dy}{y^{1/3}} = dx$$

$$\Rightarrow \frac{y^{2/3}}{(2/3)} = x + C$$

$$\Rightarrow \frac{3}{2} y^{2/3} = x + C \Rightarrow y^{\frac{2}{3}} = \frac{2}{3}(x+C)$$

Now $y(0) = 0 \Rightarrow 0 = \frac{2}{3}(0+C) \Rightarrow C = 0$

$$\therefore \boxed{y^{\frac{2}{3}} = \frac{2x}{3}}$$

Solution 1

$$y = \left(\frac{2x}{3}\right)^{3/2}$$

Solution 2

$$y = -\left(\frac{2x}{3}\right)^{3/2}$$

Solution 3

$$y = 0$$

We have three solutions to the above equation.

All the three solutions satisfy the initial condition $y(0) = 0$.

Therefore, the solution is <u>not</u> unique.

Interval of Existence: Sometimes, the solution may exist only on a small portion of the real line. And the interval itself could depend on the initial condition.

Ex: $y' = y^2$
$y(0) = y_0 \quad ; \quad y_0 \neq 0$

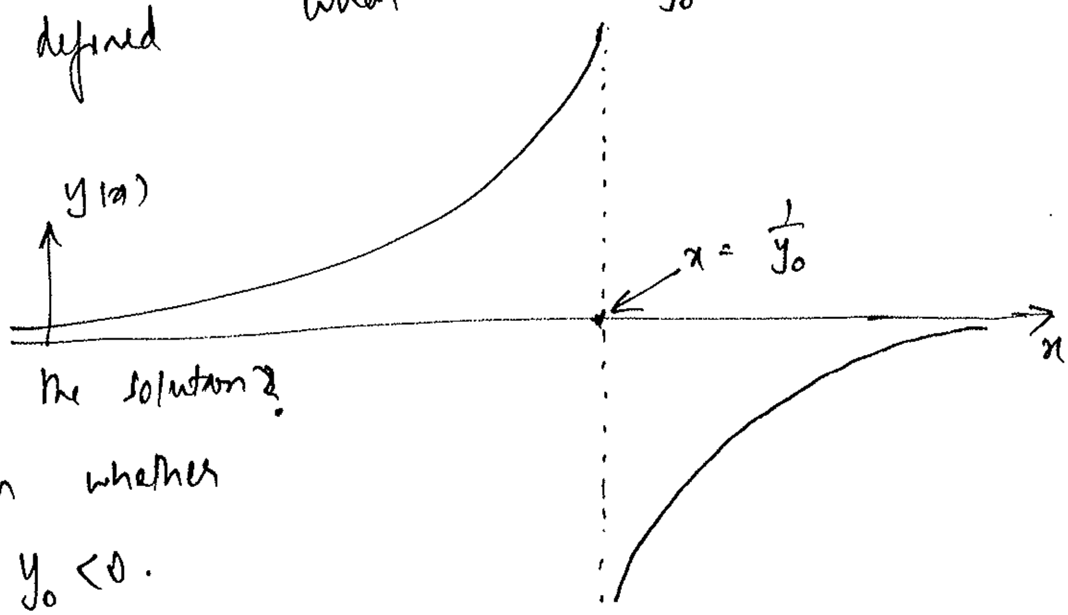
(A) $\dfrac{dy}{y^2} = dx \Rightarrow \dfrac{-1}{y} = x + c$

$\Rightarrow y = \dfrac{-1}{x+c}$

Now, $y(0) = y_0 \Rightarrow y_0 = \dfrac{-1}{c} \Rightarrow c = \dfrac{-1}{y_0}$

$\therefore \; y(x) = \dfrac{-1}{x - \frac{1}{y_0}} = \dfrac{y_0}{1 - x y_0}$

$y(x)$ is not defined when $x = \dfrac{1}{y_0}$




We have two branches.

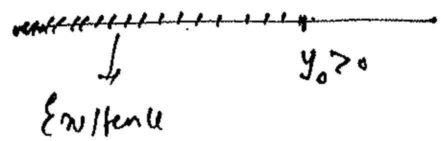
Which branch is the solution?

This depends on whether $y_0 > 0$ or $y_0 < 0$.

Interval of Existence:

→ If $y_0 > 0$, then the initial condition $y(0) = y_0$ is satisfied for the left branch

$$\Rightarrow \quad -\infty < x < \frac{1}{y_0}$$




→ If $y_0 < 0$, then initial condition is satisfied for the right branch

$$\Rightarrow \quad \frac{1}{y_0} < x < \infty$$