Math 215/255: Elementary Differential Equations I

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First Order Equations

Linear Equations

$y' + p(x)y = q(x)$

- Write the equation in the standard form,
- Calculate integrating factor, $\phi = e^{\int p dx}$,
- Multiply equation with ϕ,
- Combine first two terms using product rule
- Integrate both sides
- Calculate constant from initial condition

Separable Equations

Verify if x and y terms can be separated. If yes,

- Separate x and y terms, $f(x)dx = g(y)dy$,
- Integrate both sides

Exact Equations

$M(x, y)dx + N(x, y)dy = 0.$

- Write the equation in the standard form,
- Verify whether $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$
- Define a function which satisfies $\frac{\partial \psi}{\partial x} = M$ and $\frac{\partial \psi}{\partial y} = N$.
- Integrate the first relation and obtain ψ along with an integration constant $h(y)$
- Substitute ψ into the second relation to get $h(y)$
- Put everything together. Now ψ=constant is the solution.

Second Order Equations - Homogeneous equations

$$ay'' + by' + cy = 0$$

Put $y = e^{rt}$

$$ar^2 + br + c = 0$$

Real & distinct roots

If two roots are r_1 and r_2, then

- Fundamental solutions: $y_1 = e^{r_1 t}$, $y_2 = e^{r_2 t}$.
- General solution: $y(t) = c_1 y_1 + c_2 y_2$.

Complex roots

If $r = \lambda \pm i\mu$, then

- Fundamental solutions: $y_1 = e^{\lambda t}\cos(\mu t)$, $y_2 = e^{\lambda t}\sin(\mu t)$.
- General solution: $y(t) = c_1 y_1 + c_2 y_2$.

Real & equal roots

If $r_1 = r_2 = r$, then we have only one solution.

- First solution: $y_1 = e^{rt}$.
- Using reduction of order, we try a second solution in the form $y_2 = v(t) y_1$.
- Obtain an equation for the variable $v'(t)$. Solve for $v(t)$, and this gives $y_2(t)$.
- General solution: $y(t) = c_1 y_1 + c_2 y_2$.

Method of Undetermined coefficients - finding particular solution

$$ay'' + by' + cy = g(t)$$

$g(t)$ differs from $y_H(t)$

- **Case I:** If $g(t) = e^{\alpha t}$, then try $y_p = Ae^{\alpha t}$, substitute y_p and compare $e^{\alpha t}$ terms on both sides to find A.
- **Case II:** If $g(t) = \cos \alpha t,\ \sin(\alpha t)$ or a combination of the two, then try $y_p = A\cos \alpha t + B\sin(\alpha t)$, substitute y_p and compare $\underline{\sin}$ and $\underline{\cos}$ terms on both sides to find A and B.
- **Case III:** If $g(t) = t^n$ - a polynomial, then try $y_p = At^n + Bt^{n-1} + ...$, substitute y_p and compare similar terms on both sides to find the constants.

$g(t)$ same as $y_H(t)$

- **Case I:** If $g(t) = e^{\alpha t}$, then try $y_p = Ate^{\alpha t}$, substitute y_p and compare $e^{\alpha t}$ and $te^{\alpha t}$ terms on both sides to find A.
- **Case II:** If $g(t) = \cos \alpha t,\ \sin(\alpha t)$ or a combination of the two, then try $y_p = t(A\cos \alpha t + B\sin(\alpha t))$, substitute y_p and compare $\underline{\sin}$, $\underline{\cos}$, $\underline{t\sin}$ and $\underline{t\cos}$ terms on both sides to find A and B.
- **Case III:** If $g(t) = t^n$ - a polynomial, then try $y_p = t(At^n + Bt^{n-1} + ...)$, substitute y_p and compare similar terms on both sides to find the constants.

Variation of parameters - finding particular solution

$$y'' + p(t)y' + q(t)y = g(t)$$

If y_1 and y_2 are homogeneous solutions, then $y_H(t) = c_1 y_1(t) + c_2 y_2(t)$

- Particular solution: Put $y = u_1(t)y_1 + u_2(t)y_2$.
- Obtain y', set $u_1' y_1 + u_2' y_2 = 0$ - Eq (1).
- From the simplified y', calculate y'' to obtain a second equation between u_1' and u_2' - Eq (2).
- Solve for the variables u_1' and u_2'. Integrate these to get u_1 and u_2.
- Put everything together to get the particular solution.

Laplace Transforms

$$ay'' + by' + cy = g(t)$$

Useful things to remember:

- $F(s) = \mathcal{L}\{f(t)\} = \int_0^\infty e^{-st} f(t)dt.$
- If $\mathcal{L}\{y(t)\} = Y(s)$, then $\mathcal{L}\{y'(t)\} = sY(s) - y(t = 0)$ and $\mathcal{L}\{y''(t)\} = s^2 Y(s) - sy(t = 0) - y'(t = 0)$.
- Never mix up variables s and t in the same equation.
- Unit step function:

$$u_c(t) = \begin{cases} 0 & 0 \leq t < c, \\ 1 & t \geq c. \end{cases}$$

- If $f(t) = \mathcal{L}^{-1}\{F(s)\}$, then $\mathcal{L}^{-1}\{e^{-cs}F(s)\} = u_c(t)f(t - c)$. This is available in the table.
- Where ever required, express $g(t)$ in terms of $u_c(t)$ and then modify the expression to get it in the form $u_c(t)g(t - c)$. Always check to make sure your expression for $g(t)$ is correct.

Homogeneous Linear Equations
$\mathbf{X}' = \mathbf{A}\mathbf{X}$

↓ Put $\mathbf{X} = \xi e^{\lambda t}$

$(\mathbf{A} - \lambda \mathbf{I})\xi = \mathbf{0}$

↓

Eigenvalues: $|\mathbf{A} - \lambda \mathbf{I}| = 0$

λ's: real & distinct

λ's: complex

λ's: repeated

Real & distinct eigenvalues

- Write two fundamental solutions as $\mathbf{X}^{(1)} = \xi^{(1)} e^{\lambda_1 t}$ and $\mathbf{X}^{(2)} = \xi^{(2)} e^{\lambda_2 t}$
- General solution is $\mathbf{X} = c_1 \mathbf{X}^{(1)} + c_2 \mathbf{X}^{(2)}$.

Complex eigenvalues

- Write $\mathbf{X}^{(1)} = \xi^{(1)} e^{\lambda_+ t}$
- Take real and imaginary part of $\mathbf{X}^{(1)}$
- If $\mathbf{U} = Re\{\mathbf{X}^{(1)}\}$ and $\mathbf{V} = Im\{\mathbf{X}^{(1)}\}$, then general solution is $\mathbf{X} = c_1 \mathbf{U} + c_2 \mathbf{V}$.

Repeated eigenvalues

- First solution: $\mathbf{X}^{(1)} = \xi^{(1)} e^{\lambda t}$
- Second solution: Put $\mathbf{X} = \xi t e^{\lambda t} + \eta e^{\lambda t}$
- Substitute to obtain $(\mathbf{A} - \lambda \mathbf{I})\eta = \xi$ and find η.
- Put everything together to get $\mathbf{X}^{(2)}$. General solution is $\mathbf{X} = c_1 \mathbf{X}^{(1)} + c_2 \mathbf{X}^{(2)}$.

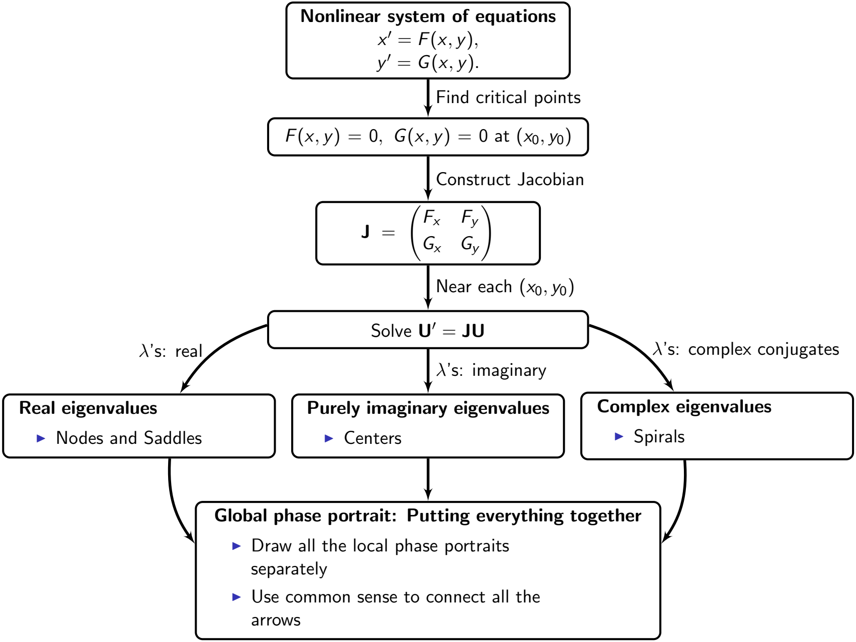
Homogeneous Linear Equations: Classification of critical point

$$\mathbf{X}' = \mathbf{AX}$$

$(0,0)$ is a critical/equilibrium point

Steps:

- Put $\mathbf{X} = \xi e^{\lambda t}$.
- Substitute to get $(\mathbf{A} - \lambda \mathbf{I})\xi = \mathbf{0}$.
- Compute eigenvalues and eigenvectors of the matrix $\mathbf{A}$.
  - If λ_1 and λ_2 are real and of same sign - Node.
  - If λ_1 and λ_2 are real and of opposite sign - Saddle.
  - If λ_1 and λ_2 are complex conjugates - Spiral. The real part of $\lambda_{1,2}$ controls the growth/decay of spiral.
  - If λ_1 and λ_2 are purely imaginary, i.e. no real part - Center.
- If $Re\{\lambda_{1,2}\} > 0$, then the critical point is unstable.
- Two solutions: $\mathbf{X}^{(1)} = \xi^{(1)} e^{\lambda_1 t}$ and $\mathbf{X}^{(2)} = \xi^{(2)} e^{\lambda_2 t}$


Nonlinear system of equations
$x' = F(x, y),$
$y' = G(x, y).$
Find critical points
$F(x, y) = 0, \; G(x, y) = 0$ at (x_0, y_0)
Construct Jacobian
$\mathbf{J} = \begin{pmatrix} F_x & F_y \\ G_x & G_y \end{pmatrix}$
Near each (x_0, y_0)
Solve $\mathbf{U}' = \mathbf{J}\mathbf{U}$
λ's: real
λ's: imaginary
λ's: complex conjugates
Real eigenvalues
- Nodes and Saddles
Purely imaginary eigenvalues
- Centers
Complex eigenvalues
- Spirals
Global phase portrait: Putting everything together
- Draw all the local phase portraits separately
- Use common sense to connect all the arrows