Complex Eigenvalues $\quad \vec{X}' = A\vec{X}$

We now consider cases when A can have complex eigenvalues.

Ex! $\vec{X}' = \underbrace{\begin{bmatrix} -\frac{1}{2} & 1 \\ -1 & -\frac{1}{2} \end{bmatrix}}_{A} \vec{X}$

We gain start with the same assumption for $\vec{X}$.

Let $\vec{X} = \vec{\xi} e^{rt}$

$\therefore \vec{X}' = \vec{\xi} r e^{rt}$

Substituting, we have $\vec{\xi} r e^{rt} = A\vec{\xi} e^{rt}$

$$\Rightarrow (A - rI)\vec{\xi} = 0$$

Eigenvalues: $|A - rI| = 0 \quad \Rightarrow \begin{vmatrix} -\frac{1}{2} - r & 1 \\ -1 & -\frac{1}{2} - r \end{vmatrix} = 0$

$$\Rightarrow \left(-\frac{1}{2} - r\right)^2 + 1 = 0$$

$$\Rightarrow \frac{1}{4} + r^2 + r + 1 = 0 \quad \Rightarrow 4r^2 + 4r + 5 = 0$$

$$\Rightarrow r = \frac{-4 \pm \sqrt{16 - 4 \cdot 4 \cdot 5}}{8} = \frac{-4 \pm \sqrt{16(-4)}}{8}$$

$$= \frac{-4 \pm 4(2i)}{4 \cdot 2} = -\frac{1}{2} \pm i$$

$\therefore r_1 = -\frac{1}{2} + i \quad ; \quad r_2 = \frac{-1}{2} - i$

Notice that the eigenvalues are complex conjugates. This will always happen if the elements of A are real numbers.

Check for correctness of Eigenvalues:

- Sum of eigenvalues $=$ Trace $(A) = \left(\frac{-1}{2}\right) + \left(\frac{-1}{2}\right) = -1$
- Product of Eigenvalues $= |A| = \frac{5}{4}$

Eigenvectors:- with $r_1 = \frac{-1}{2} + i$:

$$(A - r_1 I)\vec{\xi}^{(1)} = 0 \Rightarrow \left\{ \begin{bmatrix} \frac{-1}{2} & 1 \\ -1 & \frac{-1}{2} \end{bmatrix} - \left(\frac{-1}{2} + i\right) \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right\} \begin{bmatrix} \xi_1 \\ \xi_2 \end{bmatrix} = 0$$

$$\Rightarrow \begin{bmatrix} \frac{-1}{2} + \frac{1}{2} - i & 1 \\ -1 & \frac{-1}{2} + \frac{1}{2} - i \end{bmatrix} \begin{bmatrix} \xi_1 \\ \xi_2 \end{bmatrix} = 0$$

$$\Rightarrow \begin{bmatrix} -i & 1 \\ -1 & -i \end{bmatrix} \begin{bmatrix} \xi_1 \\ \xi_2 \end{bmatrix} = 0 \Rightarrow \left. \begin{array}{l} -i \cdot \xi_1 + \xi_2 = 0 \\ -\xi_1 - i\xi_2 = 0 \end{array} \right\} \xi_2 = i\xi_1$$

If $\xi_1 = 1,\ \xi_2 = i \Rightarrow \vec{\xi}^{(1)} = \begin{bmatrix} 1 \\ i \end{bmatrix}$

with $r_2 = \frac{-1}{2} - i$:-

$$(A - r_2 I)\vec{\xi}^{(2)} = 0 \Rightarrow \begin{bmatrix} i & 1 \\ -1 & i \end{bmatrix} \begin{bmatrix} \xi_1 \\ \xi_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow \left. \begin{array}{l} i\xi_1 + \xi_2 = 0 \\ -\xi_1 + i\xi_2 = 0 \end{array} \right\} \xi_1 = i\xi_2 \Rightarrow -i\xi_1 = \xi_2$$

If $\xi_1 = 1$, then $\xi_2 = -i$

$$\Rightarrow \quad \vec{\xi}^{(2)} = \begin{bmatrix} 1 \\ -i \end{bmatrix}$$

Again notice that $\vec{\xi}^{(1)}$ & $\vec{\xi}^{(2)}$ are complex conjugates.

In the exam, you can directly write the ~~eigen~~ second eigenvalue & eigenvector knowing that it is the complex conjugate of the first set.

<u>Solutions:</u>

$$\vec{X}^{(1)} = \begin{bmatrix} 1 \\ i \end{bmatrix} e^{\left(\frac{-1}{2}+i\right)t}$$

$$\vec{X}^{(2)} = \begin{bmatrix} 1 \\ -i \end{bmatrix} e^{\left(\frac{-1}{2}-i\right)t}$$

Since $\vec{X}^{(1)}$ & $\vec{X}^{(2)}$ are complex quantities, we will try to obtain real quantities to write the general solution.

$$\vec{X}^{(1)}(t) = \begin{bmatrix} 1 \\ i \end{bmatrix} e^{\left(\frac{-1}{2}+i\right)t} = \begin{bmatrix} 1 \\ i \end{bmatrix} e^{\frac{-1}{2}t} e^{it}$$

$$= e^{\frac{-1}{2}t} \begin{bmatrix} 1 \\ i \end{bmatrix} (\cos t + i \sin t) = e^{-\frac{t}{2}} \begin{bmatrix} \cos t + i \sin t \\ i \cos t - \sin t \end{bmatrix}$$

$$= e^{-\frac{t}{2}} \begin{bmatrix} \cos t \\ -\sin t \end{bmatrix} + i\, e^{-t/2} \begin{bmatrix} \sin t \\ \cos t \end{bmatrix}$$

$$= \vec{u} + i \vec{V},$$ $\vec{u}$ & $\vec{v}$ are real functions.

Since $\vec{X}^{(1)}(t) = \vec{u} + i\vec{v}$, We should have $\vec{X}^{(2)} = \vec{u} - i\vec{v}$.

We can obtain $\vec{X}^{(2)} = \vec{u} - i\vec{v}$ directly from $\vec{\xi}^{(2)} e^{\left(-\frac{1}{2} - it\right)}$.

General Solution:- $\vec{X} = c_1 \vec{X}^{(1)} + c_2 \vec{X}^{(2)}$

$$= c_1 (\vec{u} + i\vec{v}) + c_2 (\vec{u} - i\vec{v})$$

Combining the constants c_1 & c_2, we have

$$\vec{X} = (c_1 + c_2)\vec{u} + i(c_1 - c_2)\vec{v}$$

$$\boxed{\vec{X} = c_3 \vec{u} + c_4 \vec{v}}$$

where $c_3 = c_1 + c_2$, $c_4 = i(c_1 - c_2)$

$$\therefore \vec{X} = c_3 e^{-t/2} \begin{bmatrix} \cos t \\ -\sin t \end{bmatrix} + c_4 e^{-t/2} \begin{bmatrix} \sin t \\ \cos t \end{bmatrix}$$

Plotting the two solutions $\vec{X}^{(1)}$ & $\vec{X}^{(2)}$:-

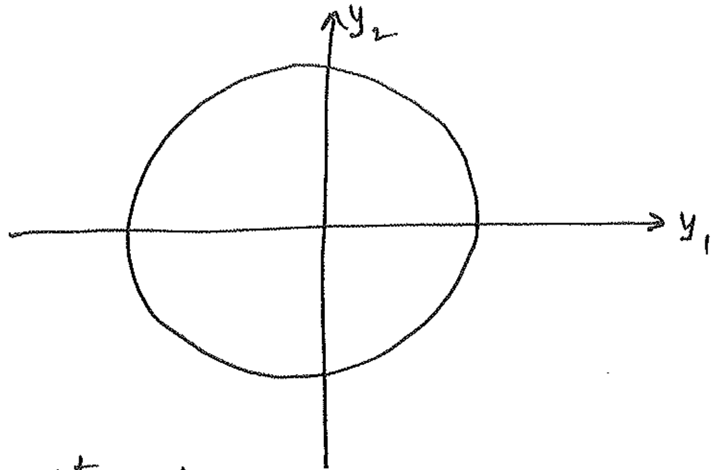
We first need to learn how to plot a trigonometric function in the 2D plane.

If $\vec{y} = \begin{bmatrix} \cos t \\ -\sin t \end{bmatrix} = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$, then $y_1 = \cos t$, $y_2 = -\sin t$

$$\Rightarrow y_1^2 + y_2^2 = \cos^2 t + \sin^2 t = 1$$

↳ Equation of a circle.

Therefore without the exponential term $\vec{y}$ is a circle.



In the presence of the e^{-t} term,

$$\vec{u} = e^{-t/2}\begin{bmatrix}\cos t\\ -\sin t\end{bmatrix}$$

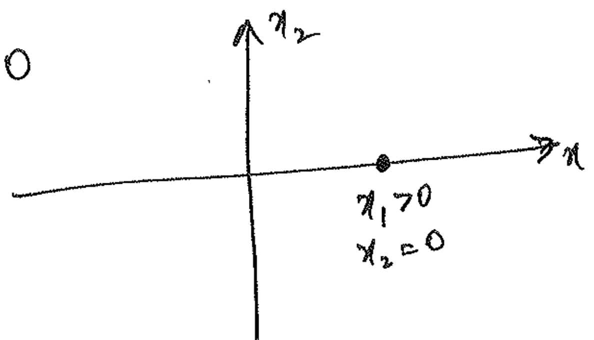
As $t \to \infty$, $\vec{u} \to 0 \Rightarrow$ the circle changes to a spiral.

How to determine direction of the spiral?

Let us examine the original equation : $\vec{x}\,' = A\vec{x}$

$$\Rightarrow \begin{bmatrix} x_1' \\ x_2' \end{bmatrix} = \begin{bmatrix} -1/2 & 1 \\ -1 & -1/2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \Rightarrow \begin{aligned} x_1' &= -\frac{x_1}{2} + x_2 \quad —(1) \\ x_2' &= -x_1 - \frac{x_2}{2} \quad —(2) \end{aligned}$$

Consider the point $x_1 > 0$ & $x_2 = 0$



From equation (2), $x_2' = -x_1 - \dfrac{x_2}{2}$

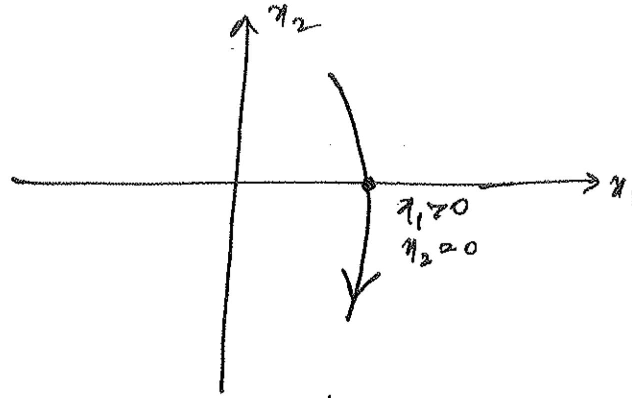
At this point, $x_2' = -x_1$

< 0 (since $x_1 > 0$)

If $x_2' < 0 \Rightarrow x_2$ decreases from this point.

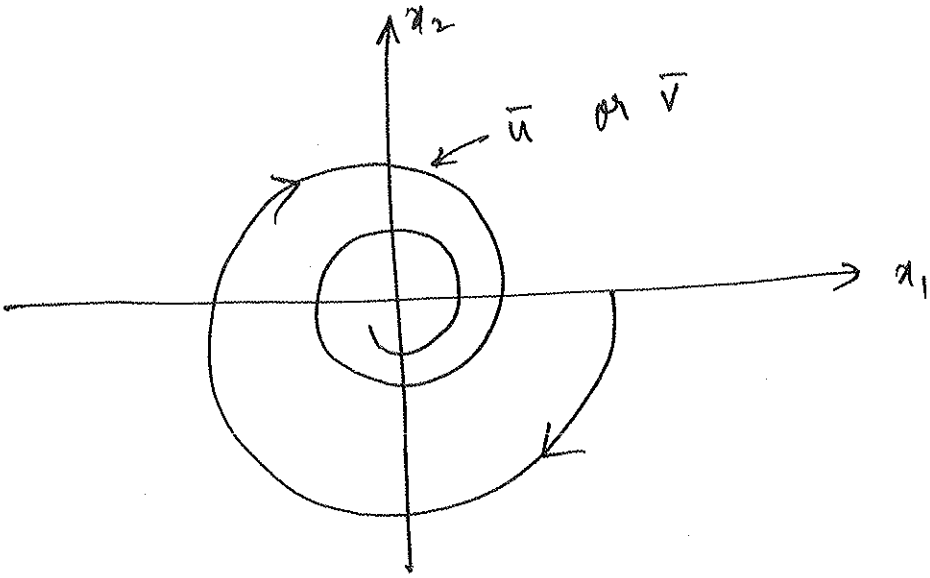
$\Rightarrow$ the curve (spiral) is in the clockwise direction.

Q.



$\therefore\ \vec{u} = e^{-t/2}\begin{bmatrix}\cos t\\ -\sin t\end{bmatrix}$ is a clockwise spiral.

Similarly $\vec{v}$ is also a clockwise spiral.



~~One more example~~ :-

Solution Procedure :- for Complex Eigenvalues

① ~~Se~~ Put $\vec{X} = \vec{\xi} e^{rt}$ in $\vec{X}' = A\vec{X}$

② Obtain Eigenvalues. These will be complex conjugates if A is a real matrix.

Eigenvalues $\longrightarrow$ $\lambda_{\pm}$

③ Using λ_+, find the eigenvector. This will be a complex eigenvector. ~~...~~

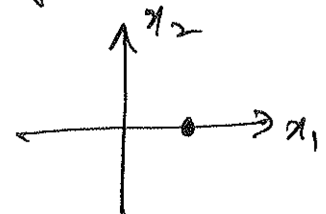
One of the solutions will be $\vec{X}^{(1)} = e^{\lambda_+ t}\vec{\xi}^{(1)}$.

④ Separate the real and imaginary parts of $\vec{X}^{(1)}$.

ie; $\vec{X}^{(1)} = \vec{X}_R + i\vec{X}_I$ ($\vec{X}_R$ is same as $\vec{u}$ & $\vec{X}_I$ is same as $\vec{v}$)

⑤ General solution, $\boxed{\vec{X} = C_1\vec{X}_R + C_2\vec{X}_I}$

Plotting: Choose a point on one of the axes, say for $x_1 > 0$ & $x_2 = 0$.



from the equation $\vec{X}' = A\vec{X}$, obtain the equation for the second component, ie; $x_2' = \ldots$

If $x_2' > 0 \Rightarrow x_2$ increases $\Longrightarrow$

If $x_2' < 0 \Rightarrow x_2$ decreases $\Longrightarrow$