CLASSIFICATION OF DIFFERENTIAL EQUATIONS

There are four ways to classify a differential equation, and these fall into the following categories:

① Order
② Linear / Non-linear
③ Homogeneous / Inhomogeneous
④ Ordinary or Partial Differential Equation.

① **Order**:- The order of a differential equation (DE) is the order of the highest derivative contained in it.

Ex: (i) $\frac{dy}{dx} = ax + b$: This is a 1st order DE since the highest derivative is $\left(\frac{dy}{dx}\right)$

(ii) $y^2\left(\frac{dy}{dx}\right)^2 + 3\frac{d^2y}{dx^2} + 4x = 0$: This is a 2nd order DE since the highest derivative is 2nd order: $\frac{d^2y}{dx^2}$

(iii) $\left(\frac{d^2y}{dx^2}\right)\left(\frac{dy}{dx}\right) + 3\left(\frac{dy}{dx}\right)^2 = 12x^3$: 2nd order since the highest derivative is $\frac{d^2y}{dx^2}$

② **Linear/Non linear**:

If y is the dependent variable, then a linear DE does not contain any term involving product of y with itself, or y with its derivatives, or product of derivative.

Everything else is non-linear.

Ex: (i) $\frac{dy}{dx} = ax + b$: Linear

(ii) $y^2 \left(\frac{dy}{dx}\right)^2 + 3\frac{d^2y}{dx^2} + 4x = 0$: Nonlinear

product of y with derivatives. Also nonlinear because of y^2 & $\left(\frac{dy}{dx}\right)^2$ itself.

(iii) $\frac{dy}{dx} = 3y^2 + 2$: Nonlinear

↳ product of y with itself.

(iv) $\frac{d^2y}{dx^2} + 3\frac{dy}{dx} = e^y$: Nonlinear

↳ This is a nonlinear term in y since it involves terms containing products of y with itself.

(3) Homogeneous / Inhomogeneous:

In general, an n^{th} order linear differential equation will be of the form:

$$a_n(x)\frac{d^ny}{dx^n} + a_{n-1}(x)\frac{d^{n-1}y}{dx^{n-1}} + \cdots\cdots + a_2(x)\frac{d^2y}{dx^2} + a_1(x)\frac{dy}{dx} + a_0(x)y(x) = C(x)$$

↳ This DE is a linear combination of all possible derivatives, hence the most general linear DE of n^{th} order.

(3) Homogeneous / Inhomogeneous :-

A differential equation is said to be inhomogeneous if it contains atleast one term with just the independent variables or with constants. ~~If~~ On the other hand, if every term involves the dependent variable, then it is a homogeneous equation.

Revisiting the examples before:

(i) $\frac{dy}{dx} = ax + b$: Inhomogeneous

Contains two terms, one with just the independent variable, & one with a constant.

(ii) $y^2\left(\frac{dy}{dx}\right)^2 + 3\frac{d^2y}{dx^2} + 4x = 0$: Inhomogeneous
↳ Contains just the independent variable.

(iii) $\frac{d^2y}{dx^2} + 3\frac{dy}{dx} + 4y = 0$: Homogeneous. Every term contains y, the dependent variable.

(4) Ordinary / Partial Differential Equation :-

A differential equation involves derivatives of a ~~function~~ function (the dependent variable). But, if there is more than one independent variable, then we will have to deal with derivatives with respect to ~~both~~ all the independent variables.

The differential equation for the function $y(x)$ can only involve derivatives of y with respect to x, ex: $\frac{dy}{dx}, \frac{d^2y}{dx^2}, \ldots$

But a differential equation for $y(x,t)$ will involves derivatives of y with respect to both x & t, ie; $\frac{\partial y}{\partial x}, \frac{\partial y}{\partial t}, \frac{\partial^2 y}{\partial x \partial t}$, etc.. These are called "Partial Derivatives", & such differential equations are called Partial Differential Equations. (PDE).

If we have only one independent variable, then we get Ordinary Differential Equations (ODE).

Famous equation from physics:-

(i) Wave equation : $\frac{\partial^2 y}{\partial t^2} = c^2 \frac{\partial^2 y}{\partial x^2}$ (Also linear)

↳ A good model to describe 1D waves for a variety of problems ranging from waves on a string to ocean waves.

Multi-dimensional wave equation: $\frac{\partial^2 u}{\partial t^2} = c^2 \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} \right)$

• Nonlinear wave equation: $\frac{\partial^2 u}{\partial t^2} = c(u) \frac{\partial^2 u}{\partial x^2}$

Since $c(u)$ is a function of u, we will have product of u with its derivatives.

(ii) Heat Equation: $\frac{\partial y}{\partial t} = \alpha \frac{\partial^2 y}{\partial x^2}$: Linear PDE