28/March/2012

AUTONOMOUS SYSTEMS AND STABILITY

$$\frac{dx}{dt} = F(x,y)$$
$$\frac{dy}{dt} = G(x,y)$$

F and G are continuous and have continuous partial derivative in some domain D.

$$\vec{X} = x\hat{i} + y\hat{j}$$
$$\vec{f}(\vec{x}) = F(x,y)\,\hat{i} + G(x,y)\,\hat{j}$$

If $\vec{X}(t_0) = \vec{X}^{(0)}$, then

$$\boxed{\frac{d\vec{X}}{dt} = \vec{f}(\vec{X}) \quad \text{with} \quad \vec{X}(t_0) = \vec{X}_0}$$

F and G do not depend on independent variable t

$\Rightarrow$ Autonomous System.

$\boxed{\vec{X}' = A\vec{X}}$: Example of Autonomous System.

$\vec{f}(\vec{X}) = 0$: Gives us critical points.

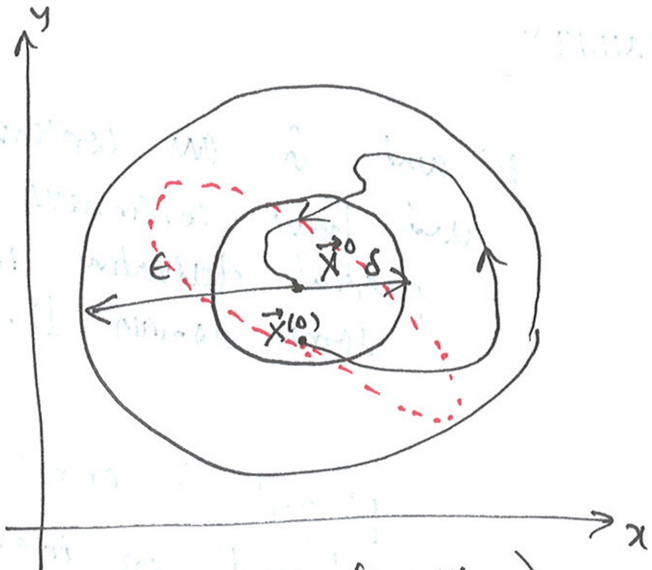
At these points $\frac{d\vec{X}}{dt} = 0$

Stability of a critical point :- $[\vec{X}^{(0)}]$

stable if for any $\epsilon > 0$, $\exists$ $\delta > 0$ such that every solution $\vec{X}(t)$ which satisfies at $t=0$

$$\left\| \vec{X}(0) - \vec{X}^{0} \right\| < \delta$$

exists and satisfies $\left\| \vec{X}(t) - \vec{X}^{0} \right\| < \epsilon$



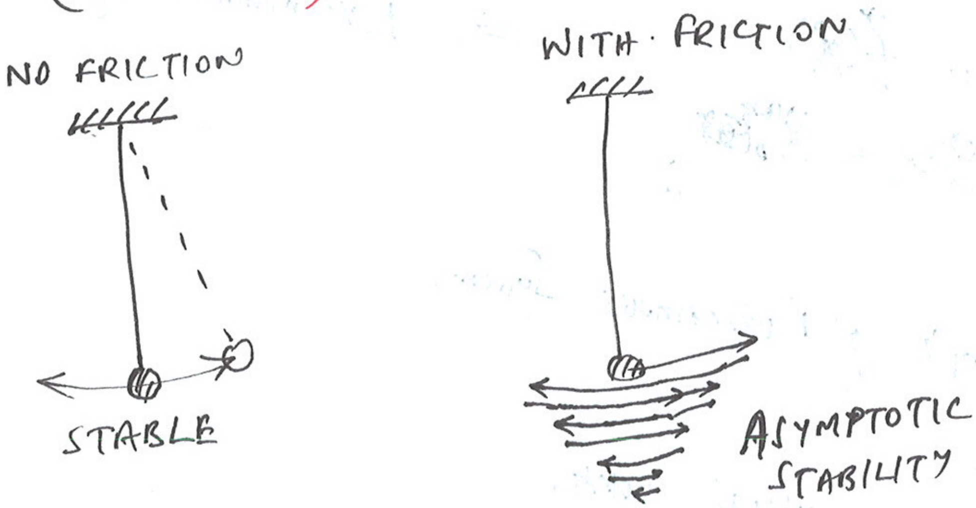
(— Asymptotic Stability.)

(.... Stable)

Asymptotic Stability:

For $\|\vec{X}(t=0) - \vec{X}^0\| < \delta$,

$$\lim_{t\to\infty} \vec{X}(t) \to \vec{X}^0$$



Linear Systems: $\vec{X}' = A\vec{X}$: $\vec{X} = 0$ is a critical point

Nonlinear Systems: $\vec{X}' = \vec{f}(\vec{X})$: $\vec{f}(\vec{X}) = 0$ can give multiple critical points.

Ex:

$$\frac{dx}{dt} = -(x-y)(1-x-y) = f(x,y)$$

$$\frac{dy}{dt} = x(2+y) = G(x,y)$$

Critical points:

$$\left.\begin{aligned} (x-y)(1-x-y) &= 0 \quad -① \\ x(2+y) &= 0 \quad -② \end{aligned}\right\}$$

$x = 0 \Rightarrow -y(1-y) = 0 \Rightarrow y = 0;\ y = 1$: $(0,0)$, $(0,1)$

$y = -2 \Rightarrow (x+2)(1-x+2) = 0$

$\Rightarrow (x+2)(3-x) = 0 \Rightarrow x = -2$ (or) $x = 3$ $\Big\}$ $(-2,-2)$, $(3,-2)$

four critical points: $(0,0)$, $(0,1)$, $(-2,-2)$, $(3,-2)$

Eigenvalues of $\vec{X}' = A\vec{X}$

$$|A - rI| = 0$$

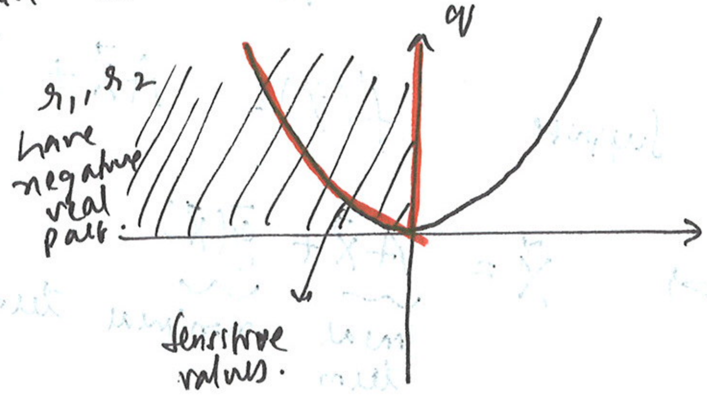
$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \qquad A + \epsilon = \begin{bmatrix} a+\epsilon_1 & b+\epsilon_2 \\ c+\epsilon_3 & d+\epsilon_4 \end{bmatrix}$$

$$|A - rI| = 0 \Rightarrow r^2 - (a+d)r + (ad-bc) = 0$$

$$r = \frac{(a+d) \pm \sqrt{(a+d)^2 - 4(ad-bc)}}{2}$$

$$|A + \epsilon - rI| = 0 \Rightarrow r = \frac{[(a+\epsilon_1)+(d+\epsilon_4)] \pm \sqrt{(a+\epsilon_1+d+\epsilon_4)^2 - 4[(a+\epsilon_1)(d+\epsilon_4) - (b+\epsilon_2)(c+\epsilon_3)]}}{2}$$

If we make small perturbation to coefficient matrix A, then this will result in a small perturbation to the eigenvalues.



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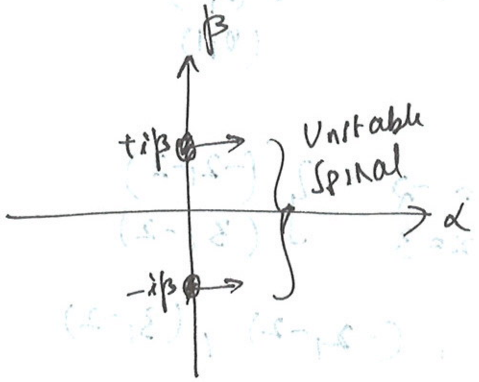
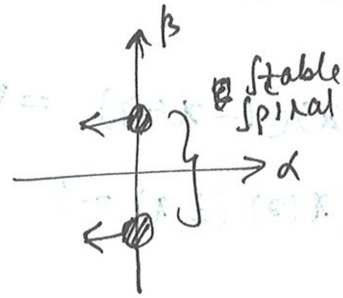
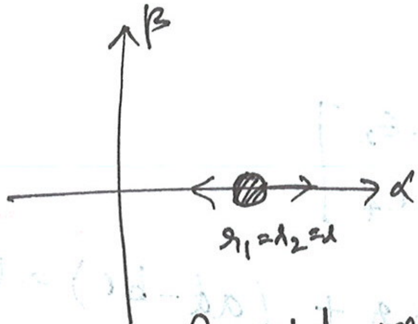
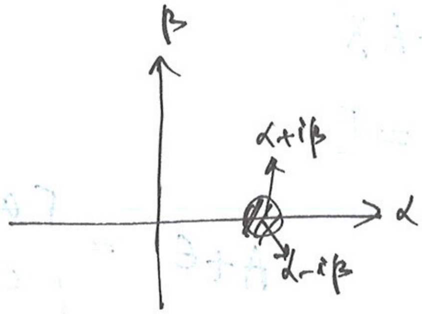




CENTER: $r = \pm i\beta$



Repeated roots



Spirals.

Locally Linear Systems:-

$\dot{\vec{X}} = \vec{f}(\vec{X})$: Nonlinear : Has multiple critical points.

$\vec{X}' = A\vec{X}$: Linear

↳ $\vec{X} = 0$: critical point (Isolated)

Suppose $\vec{f}(\vec{X}) = A\vec{X} + \vec{g}(\vec{X})$

$$\Rightarrow \quad \dot{\vec{X}} = \underbrace{A\vec{X}}_{\text{linear term}} + \underbrace{\vec{g}(\vec{X})}_{\text{nonlinear term}}$$

For the nonlinear system to be close to linear, we require

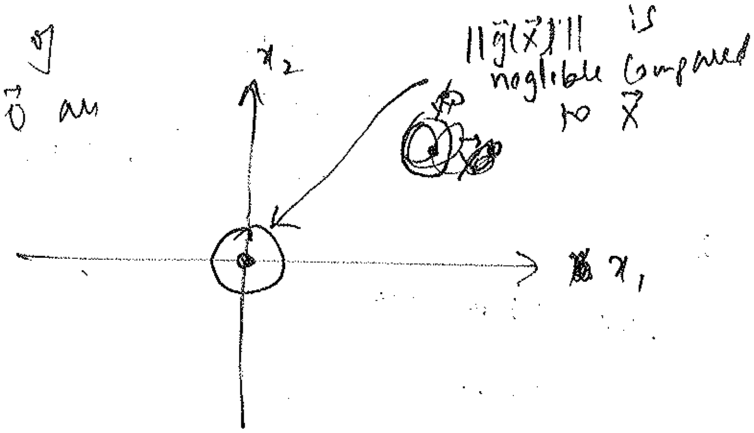
$\vec{g}(\vec{x})$ to be small.

As $\vec{X}(t)$ approaches $\vec{0}$, $\vec{g}(\vec{x})$ also approaches $\vec{0}$ but faster.

ie; $$\frac{\|\vec{g}(\vec{X})\|}{\|\vec{X}\|} \to 0 \quad \text{as } \vec{X} \to \vec{0}$$

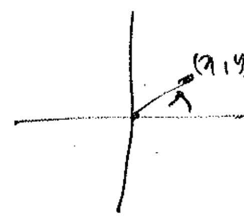
ie; the dominant terms of the equation near $\vec{X} = \vec{0}$ are

$$\underline{\underline{\vec{X}' = A\vec{X}}}$$



Such a system is called locally linear.

Let $\vec{X} = \begin{bmatrix} x \\ y \end{bmatrix}$; $\|\vec{X}\| = \sqrt{x^2+y^2} = r$



$$\vec{g}(\vec{x}) = \begin{bmatrix} g_1(x,y) \\ g_2(x,y) \end{bmatrix}$$

$$\|\vec{g}(\vec{x})\| = \sqrt{g_1^2+g_2^2}$$

$$\frac{\sqrt{g_1^2+g_2^2}}{r} \to 0 \quad \text{as } r \to 0$$

$\Rightarrow$ $\frac{g_1(x,y)}{r} \to 0$ and $\frac{g_2(x,y)}{r} \to 0$ as $r \to 0$.

<u>SKIP THIS</u>

<u>ex:</u>
$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} -x^2-xy \\ -2xy-y^2 \end{bmatrix}$$

<u>Critical points:</u>

$$x - x^2 - xy = 0 \Rightarrow x(1-x-y) = 0$$
$$2y - 2xy - y^2 = 0 \Rightarrow y(2-2x-y) = 0$$

$x(1-x-y)=0 \longrightarrow x=0$

$y(2-2x-y)=0 \longrightarrow y=0$

$\hookrightarrow x(1-x)=0 \Rightarrow x=1$ or $x=0$

$\Rightarrow (0,0),(0,1)$

$y(2-y)=0 \Rightarrow y=0$ or $y=2$

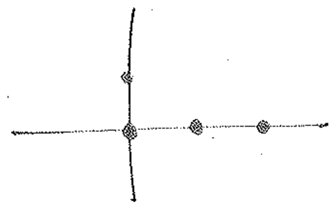
$\Rightarrow (0,0),(0,2)$

$$\begin{array}{l} 1-x-y=0 \\ 2-2x-y=0 \\ \hline 1-x=0 \end{array} \Rightarrow x=1 \Rightarrow \left.\begin{array}{l} 1-1-y=0 \\ \Rightarrow y=0 \end{array}\right\} (1,0)$$

Four critical points.

$(0,0)$, $(0,1)$, $(0,2)$

$(1,0)$

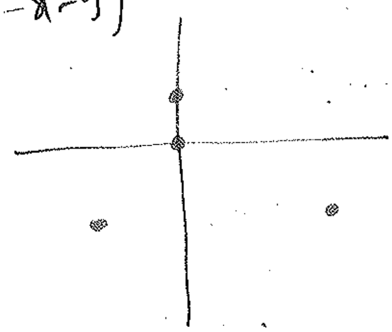


Ex 2:

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{bmatrix} -1 & 1 \\ 2 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} x^2-y^2 \\ xy \end{bmatrix}$$

$$\begin{aligned} x' = -x+y+x^2-y^2 &= -x+y+(x+y)(x-y) \\ &= -(x-y)+(x+y)(x-y) \\ &= -(x-y)\left[1-x-y\right] \end{aligned}$$

$y' = 2x+xy = x(2+y)$



Critical points:

$(0,0)$, $(0,1)$, $(-2,-2)$, $(3,-2)$

Can we show that the system is locally linear near $(0,0)$?

$\cancel{\begin{pmatrix} x' \\ y' \end{pmatrix}} = \cancel{A\vec{x}}$ $\quad \vec{X}' = A\vec{X} + \vec{g}(\vec{X})$ where

$g_1(x,y) = x^2-y^2$

$g_2(x,y) = xy$

(4)

$(0,0)$ is an isolated critical point.

Let $x = r\cos\theta$
$y = r\sin\theta$

$$\frac{g_1(x,y)}{r} = \frac{r^2\cos^2\theta - r^2\sin\theta}{r} = \frac{r^2(\cos^2\theta - \sin^2\theta)}{r} = \cancel{\cos 2\theta}$$
$$= r(\cos^2\theta - \sin^2\theta) \to 0 \text{ as } r \to 0.$$

$$\frac{g_2(x,y)}{r} = \frac{xy}{r} = \frac{r^2\cos\theta\sin\theta}{r} = r\cos\theta\sin\theta \to 0 \text{ as } r \to 0.$$

$\therefore$ $(0,0)$ is locally linear.

What about other points?

$x' = F(x,y)$
$y' = G(x,y)$

Let (x_0, y_0) be a critical point.

$\Rightarrow$ $F(x_0, y_0) = 0$
$G(x_0, y_0) = 0$

Taylor expanding $F(x,y)$ & $G(x,y)$ about (x_0,y_0):

$$F(x,y) = \underbrace{F(x_0,y_0)}_{=0} + F_x(x_0,y_0)(x-x_0) + F_y(x_0,y_0)(y-y_0) + \eta_1(x,y)$$

$$G(x,y) = \underbrace{G(x_0,y_0)}_{=0} + G_x(x_0,y_0)(x-x_0) + G_y(x_0,y_0)(y-y_0) + \eta_2(x,y)$$

where $\frac{\eta_1(x,y)}{\sqrt{(x-x_0)^2+(y-y_0)^2}} \to 0$ as $(x,y) \to (x_0,y_0)$

$\frac{\eta_2(x,y)}{\sqrt{(x-x_0)^2+(y-y_0)^2}} \to 0$ as $(x,y) \to (x_0,y_0)$

$$\frac{dx}{dt} = \frac{d}{dt}(x-x_0) \quad ; \quad \frac{dy}{dt} = \frac{d}{dt}(y-y_0)$$

$$\Rightarrow \begin{bmatrix} \frac{d}{dt}(x-x_0) \\ \frac{d}{dt}(y-y_0) \end{bmatrix} = \underbrace{\begin{bmatrix} F_x(x_0,y_0) & F_y(x_0,y_0) \\ G_x(x_0,y_0) & G_y(x_0,y_0) \end{bmatrix}}\begin{bmatrix} x-x_0 \\ y-y_0 \end{bmatrix} + \begin{bmatrix} \eta_1(x,y) \\ \eta_2(x,y) \end{bmatrix}$$

Let $\vec{u} = \begin{bmatrix} x-x_0 \\ y-y_0 \end{bmatrix}$

$$\Rightarrow \frac{d\vec{u}}{dt} = J\vec{u} + \begin{bmatrix} \eta_1(x,y) \\ \eta_2(x,y) \end{bmatrix}$$

$$J = \begin{bmatrix} F_x & F_y \\ G_x & G_y \end{bmatrix}_{at\ (x_0,y_0)} \quad : \text{Jacobian matrix}$$

$$= \frac{d\vec{f}}{d\vec{x}}(\vec{x}^0)$$

Linear system near $(0,0)$:-

$F(x,y) = -x+y+x^2-y^2$

$G(x,y) = 2x+xy$

$F_x = \frac{\partial F}{\partial x} = -1+2x$

$F_y = 1-2y$

$G_x = 2+y$

$G_y = x$

$$J = \begin{bmatrix} -1 & 1 \\ 2 & 0 \end{bmatrix}_{at\ (0,0)}$$

$$\therefore \frac{d\vec{u}}{dt} = \begin{bmatrix} -1 & 1 \\ 2 & 0 \end{bmatrix}\begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$$