AUTONOMOUS EQUATIONS AND POPULATION DYNAMICS

Consider equations of the form

$$\boxed{\frac{dy}{dt} = f(y)} \qquad (1)$$

The right-hand side of this equation, ie, $f(y)$ is not an explicit function of the independent variable t. Such equations are called Autonomous Equations and are frequently ~~encounted~~ encountered when modelling populations.

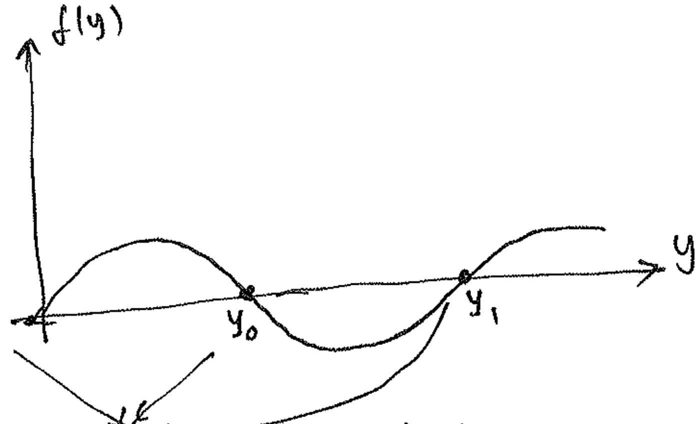
(Q) Consider the initial value problem (IVP)

$$\left.\begin{aligned} \frac{dy}{dt} &= f(y) \\ y(0) &= \alpha \end{aligned}\right\} \quad (2)$$

Find their equilibrium solutions & find out if they are stable or unstable!

→ Equilibrium or rest points are values of y for which $f(y) = 0$

Let the plot of $f(y)$ look like this!



Equilibrium values of y.

Are they stable?

- Stable equilibrium!

If $\lim_{t\to\infty} y(t) \to y_i$ for some range of initial conditions, then y_i is said to be a stable equilibrium.

GOAL: Can we draw plots of solutions (integral curves) y vs t for various initial conditions without solving the differential equation? The notes below shows how!

SIMPLE EXAMPLE:- (A brief introduction to Population Dynamics).

Let the population of a given species by $y(t)$ at any time t. If there is infinite food available & the species has no predators, then we expect the population to increase rapidly. A common model is to assume that the ~~rate~~

rate of growth at any time t $\propto$ population at the time t

At small times, this is usually a pretty decent model.

$\therefore \frac{dy}{dt} \propto y$

$\Rightarrow \boxed{\frac{dy}{dt} = ry}$ where r : rate of growth, typically positive.

Let initial population be y_0. We have an IVP ③

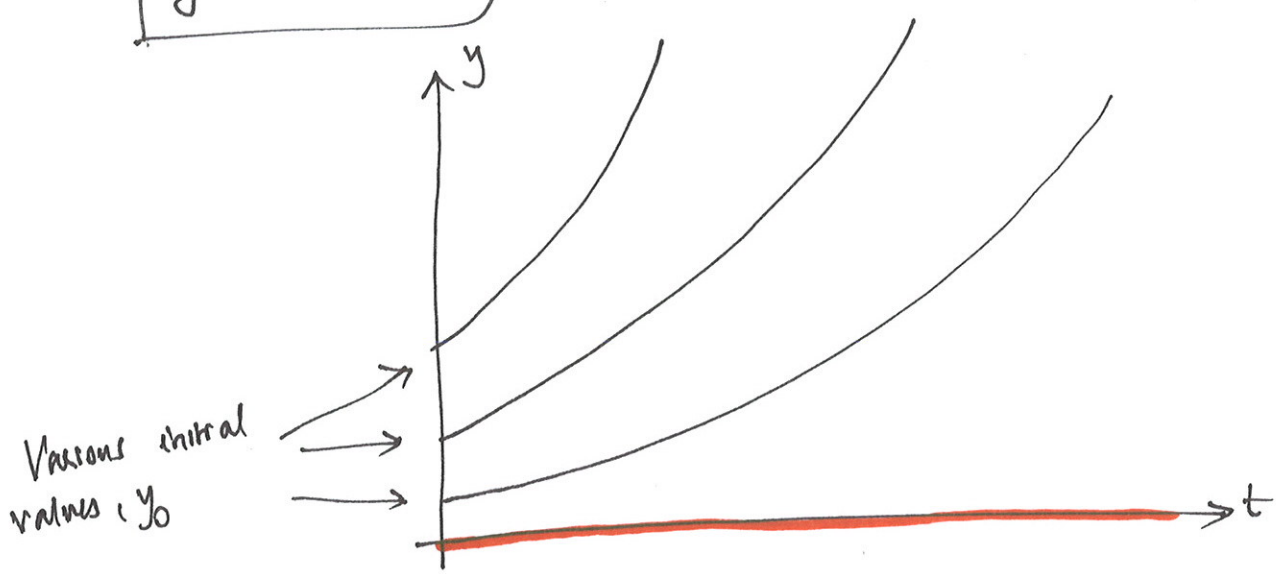
$$\boxed{\begin{aligned} \frac{dy}{dt} &= ry, \quad r > 0 \\ y(0) &= y_0 \end{aligned}} \quad \text{③}$$

Equil. Solution :-

$\frac{dy}{dt} = 0 \Rightarrow \boxed{y_{eq.} = 0}$

Solving this equation gives us

$$\boxed{y = y_0 e^{rt}} \quad \text{④}$$



The equilibrium solution $y_{eq.} = 0$ is unstable.

In reality, we usually have predators. Also, at large time, food shortages are encountered. If this happens, then it is difficult to sustain the same rate of growth.

~~At~~ ~~then~~ A simple modification to our previous model is to assume a rate of growth which ~~d~~ depends on the population itself.

At small ~~to~~ populations when there isn't much competition for resources, we expect growth rate to be constant. At very large population, we expect no growth or even a decline (i.e., negative growth rate).

<u>Model</u>:
$$\frac{dy}{dt} = r(y)\, y$$
where $r(y) = r_0 - ay$ where $r_0, a > 0$

$$\Rightarrow \quad \frac{dy}{dt} = (r_0 - ay)\, y = r_0\left(1 - \frac{a}{r_0} y\right) y$$

Let $\frac{r_0}{a} = K$. This gives us the IVP:

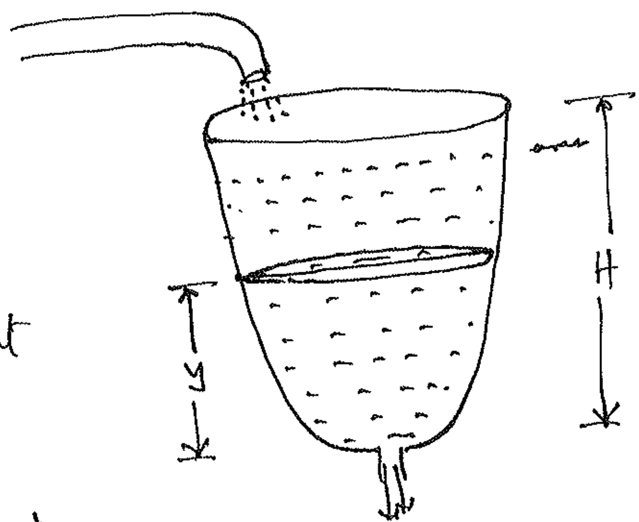
$$\boxed{\begin{aligned} \frac{dy}{dt} &= r_0\left(1 - \frac{y}{K}\right) y \\ y(0) &= y_0 \end{aligned}} \qquad (5)$$

We will analyse this equation later.

A More Complicated Example :- (LEAKEY BUCKET MODEL):

Ex: Consider water flowing into a tank or a bucket at a rate $r\ \left(\frac{GAL}{MIN}\right)$ and leaks out from a small hole at the bottom with cross-sectional area A.

Under what conditions can we prevent an overflow?



Let H be the maximum height of the tank, and $y(t)$ be the height ~~any~~ at any given time.

Also, let initial height of the fluid by y_0.

$\therefore\ y(0) = y_0$

NOTE: The density of water is fixed. ~~Hence~~ Since Mass = Density × Volume, Calculating mass flow balance is equivalent to volume flow balance.

MASS BALANCE EQUATION :-

RATE AT WHICH FLUID COMES IN $= r$ gal/min.

RATE AT WHICH FLUID LEAKS OUT $\propto A \cdot \sqrt{2gy}$

RATE AT WHICH (Since at any time, height of water is $y(t)$, velocity at the bottom from Torricelli's principle is

$$V_{exit} = \sqrt{2gy}\ )$$

$\therefore$ RATE AT WHICH FLUID LEAKS $= \alpha A \sqrt{2gy}$

where α is a proportionality constant.

$$\boxed{\text{Net Mass FLOW RATE} = \text{RATE IN} - \text{RATE OUT}} \quad (6)$$

$$= r - \alpha A \sqrt{2gy}$$

What is the net mass flow rate at any time t ?

Let $B(y)$ be the cross-sectional area of the bucket.

The rate at which the water level changes with time is $\left(\frac{dy}{dt}\right)$ since y is the water level position.

$$\therefore \boxed{\text{Net mass flow rate} = \text{Area} \times \text{Velocity}} \quad (7)$$

$$= B(y) \times \frac{dy}{dt}$$

$\therefore$ By mass conservation,

$$B(y) \frac{dy}{dt} = r - \alpha A \sqrt{2gy}$$

$$\Rightarrow \boxed{\frac{dy}{dt} = \frac{1}{B(y)} \left[r - \alpha A \sqrt{2gy} \right]} \quad (8)$$

$$\boxed{y(0) = y_0}$$

$B(y)$: A known cross-sectional area of the bucket/tank.

Equilibrium value of y :-

$\frac{dy}{dt} = 0$ [The position of water level ~~is~~ remains unaltered]

$$\Rightarrow \quad r - \alpha A \sqrt{2 g y_{eq}} = 0$$

where y_{eq} is the equilibrium height.

Solving for y_{eq} : $\boxed{y_{eq} = \frac{r^2}{2g} \frac{1}{\alpha^2 A^2}}$ —— (9)

Is this stable or unstable ?

Now if $0 < y_{eq} < H$, we require

$$\frac{r^2}{2g} \frac{1}{\alpha^2 A^2} < H$$

$$\Rightarrow \quad r^2 < 2g \alpha^2 A^2 H$$

Does $\lim_{t \to \infty} y(t) = y_{eq}$?

for any initial $y_0 \in [0, H]$, we require

$\boxed{r < (2g)^{1/2} \alpha A H^{1/2}}$ for equilibrium level to be below H.

For model (5) on page (4) & model (8) on page (5) of this notes, we will now sketch the solutions using a new technique.

Let us consider the population model introduced earlier.

$$\frac{dy}{dt} = r_0 \left(1 - \frac{y}{K}\right) y$$

$$y(0) = y_0$$

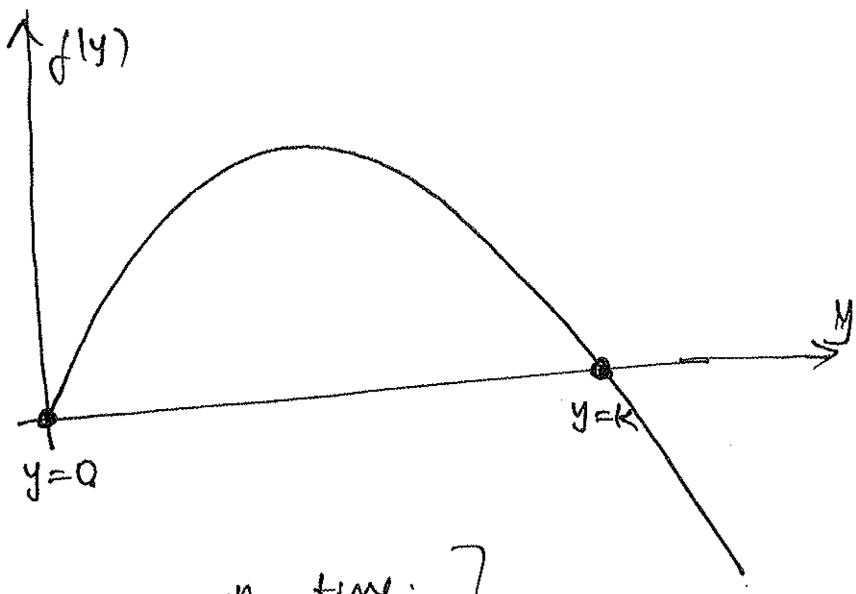
This is in the standard form $\frac{dy}{dt} = f(y)$; $y(0) = y_0$

<u>Equilibrium</u> <u>Solutions</u>: $\frac{dy}{dt} = 0$ (ie; population does not change)

$$\Rightarrow \quad f(y) = 0$$

$$\therefore \quad y = 0 \quad \text{or} \quad y = K$$

Let us look at the plot of $f(y)$. The equilibrium solutions are the roots of $f(y)$ and are shown will filled dots.



When $f(y) > 0$,

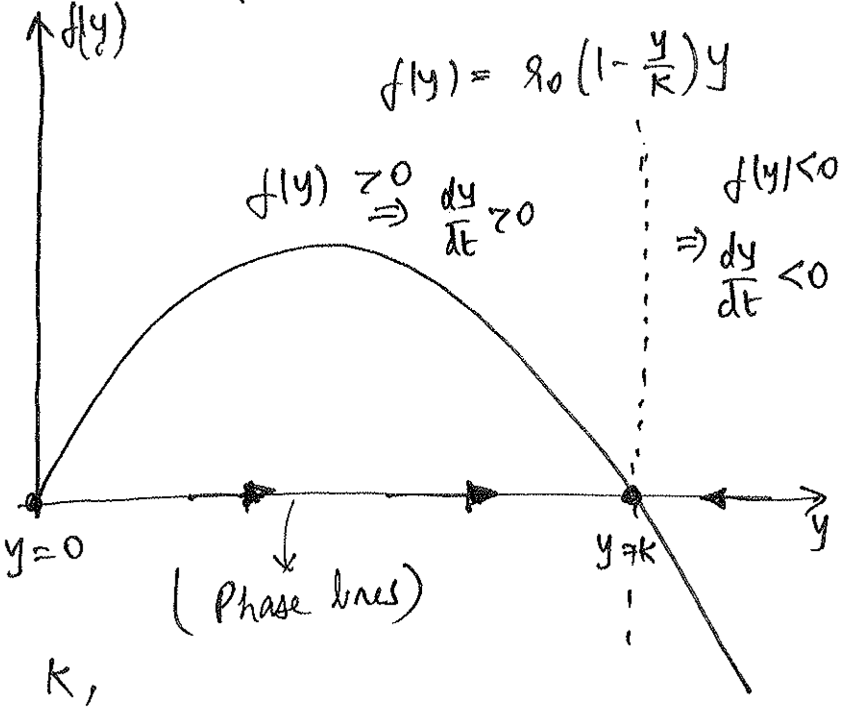
$\frac{dy}{dt} > 0 \Rightarrow y$ increases with time.

When $f(y) < 0$;

$\frac{dy}{dt} < 0 \Rightarrow y$ decreases with time.

These are shown using arrows on y-axis in the next plot.

Therefore, if our initial population $y_0 \in (0, K)$, the population increases toward $y = K$ as shown by the arrows.



Similarly, if our initial population is greater than K, i.e; $y_0 > K$, then population decreases toward $y = K$ as shown by the arrows

Therefore, $y = 0$ is an unstable equilibrium, and $y = K$ is a stable equilibrium.

Finally, we require information on concavity of the solutions to complete the analysis.

If $\frac{dy}{dt}$ gives us information on how y changes with t, $\frac{d^2y}{dt^2}$ gives us information on how the slope of y changes with time.

(a) If $\frac{dy}{dt} > 0$ & $\frac{d^2y}{dt^2} > 0$: y increasing with t, & slope increasing with t.
$\Rightarrow$ Concave up

(b) If $\frac{dy}{dt} > 0$ & $\frac{d^2y}{dt^2} < 0$: y increasing & slope decreasing.
$\Rightarrow$ Concave down

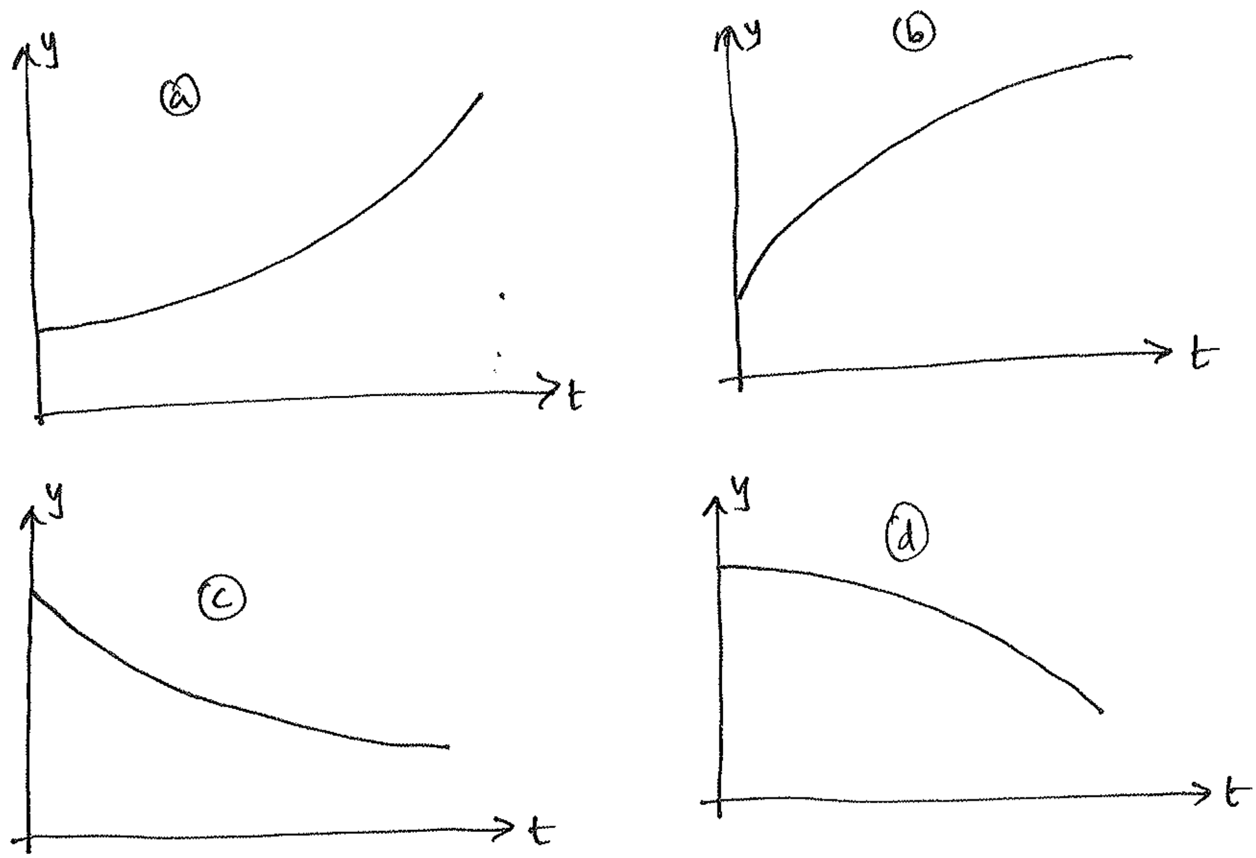
(c) If $\frac{dy}{dt} < 0$ & $\frac{d^2y}{dt^2} > 0$: y ~~slope~~ decreasing & slope increasing

$\Rightarrow$ Concave up

(d) If $\frac{dy}{dt} < 0$ & $\frac{d^2y}{dt^2} < 0$: y decreasing & slope decreasing

$\Rightarrow$ Concave down

The curves for the four cases is shown below:



Concave down

Concave up

How to determine $\frac{d^2y}{dt^2}$:-

$$\frac{d^2y}{dt^2} = \frac{d}{dt}\left(\frac{dy}{dt}\right) = \frac{d}{dt}[f(y)] = \underbrace{\frac{df}{dy}\cdot\frac{dy}{dt}}_{\text{chain rule}} = f'(y)\cdot f(y)$$

$\because \frac{dy}{dt} = f(y)$

where $f'(y) = \frac{df}{dy}$

$$\therefore \quad \frac{d^2y}{dt^2} = f'(y)\, f(y)$$

Now we have all information to plot the solution.

Population Dynamics:-

y = 0 & y = K are called the critical points.

Ex: $\dfrac{dy}{dt} = r_0\left(1 - \dfrac{y}{K}\right) y$

$y(0) = y_0$

Also $r_0, K > 0$

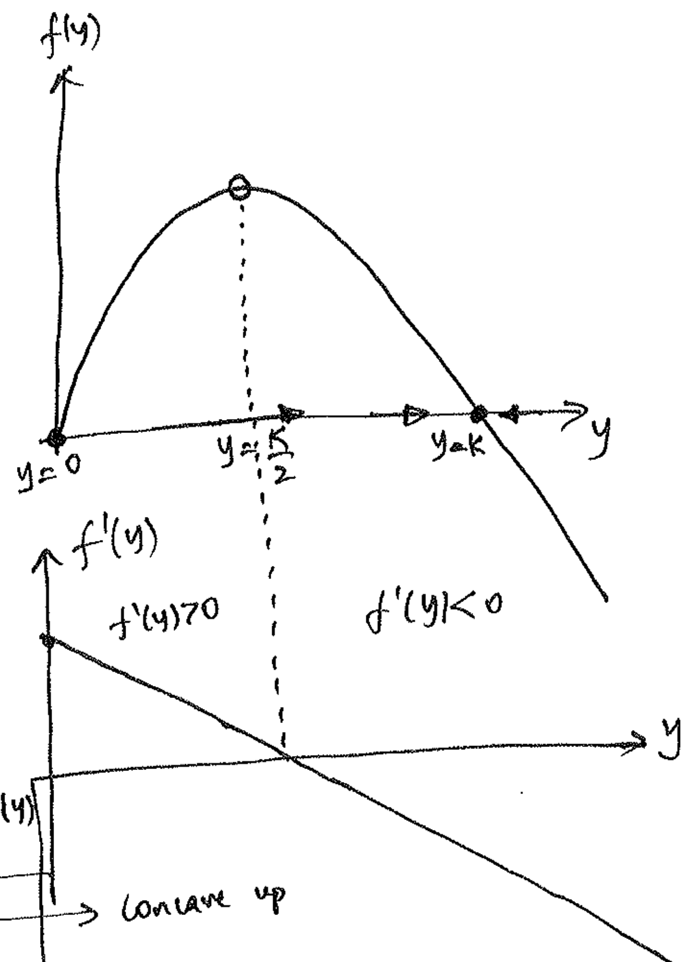
Here $f(y) = r_0\left(1 - \dfrac{y}{K}\right) y$

$$\frac{df}{dy} = r_0\left(1 - \frac{2y}{K}\right)$$

$\therefore f(y) = 0$ at $y = 0$ & K

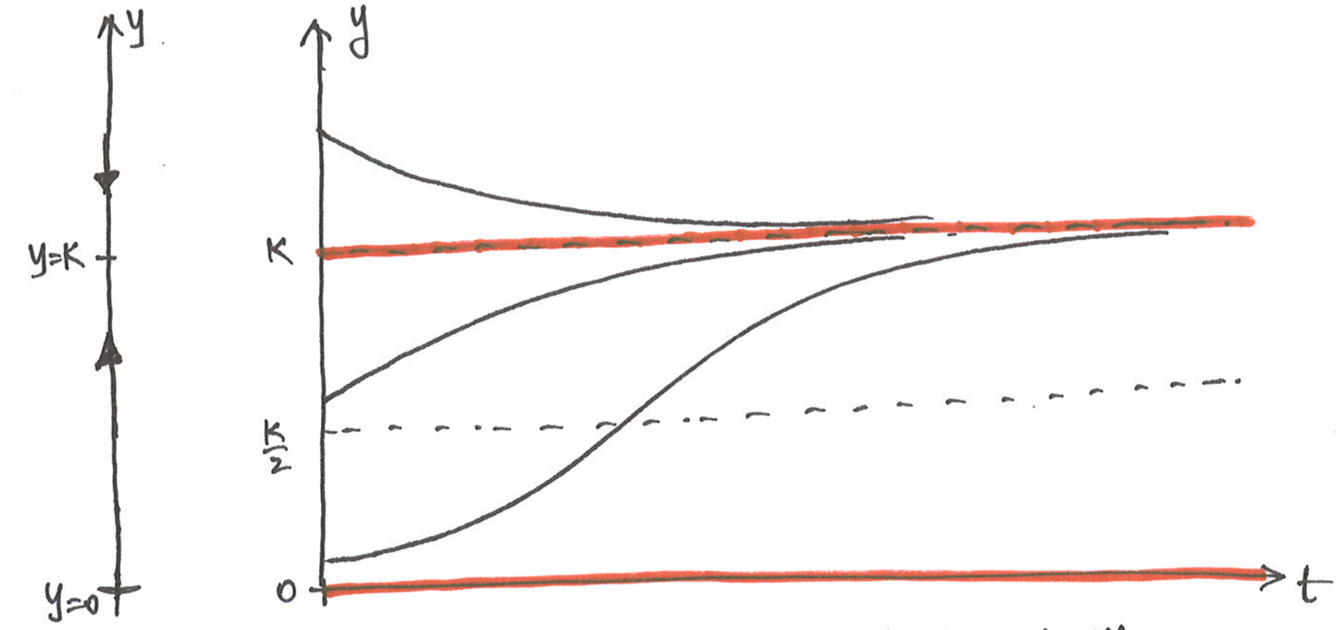
$f'(y) = 0$ at $y = \dfrac{K}{2}$

Draw plots of $f(y)$ & $f'(y)$:



Interval	$f(y) = \frac{dy}{dt}$	$f'(y)$	$\frac{d^2y}{dt^2} = f(y)\cdot f'(y)$	
$\left[0, \frac{K}{2}\right]$	>0	>0	>0	→ concave up
$\left[\frac{K}{2}, K\right]$	>0	<0	<0	→ concave down
$[K, \infty)$	<0	<0	>0	→ concave up

Note that in this problem, there are only three intervals of interest.



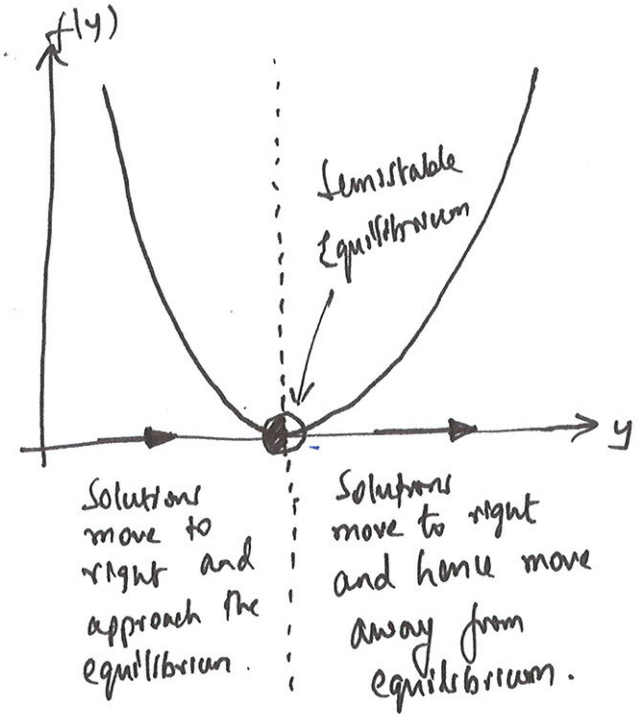
phase lines

Integral or solution curves for various initial conditions at $t=0$

SEMISTABLE EQUILIBRIUM:-

There are cases when solutions approach an equilibrium from one side, but depart away from the equilibrium from the other side. Such an equilibrium is said to be **SEMISTABLE**.

Consider the graph of $f(y)$ as shown:



Equilibrium: $f(y) = 0$

Ex: $\dfrac{dy}{dt} = y^2(y^2-1)$

Here $f(y) = y^2(y^2-1)$

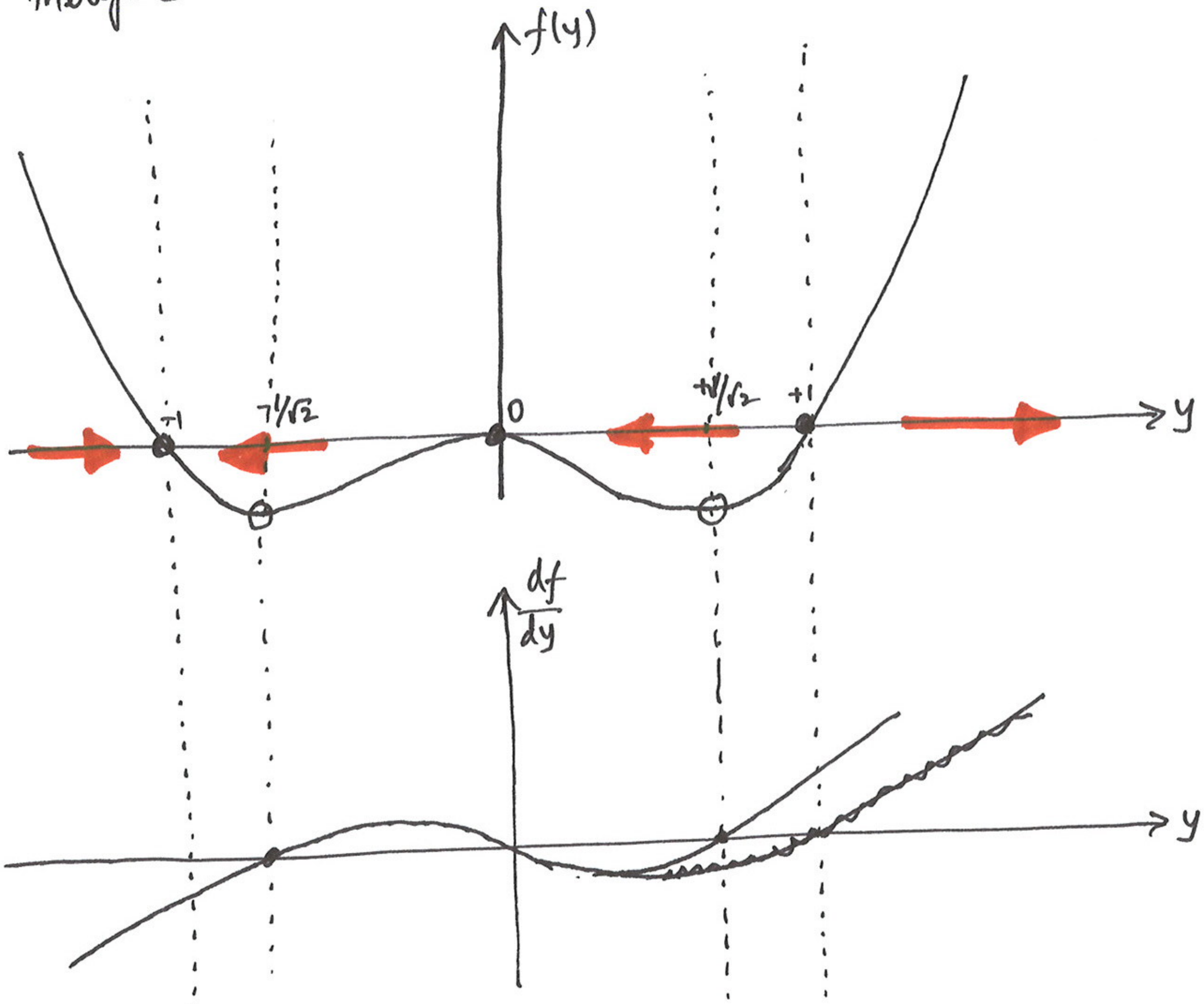
Equilibrium solutions: $f(y) = 0 \Rightarrow y = 0$ and $y = \pm 1$

We have three equilibrium solutions.

$\dfrac{df}{dy} = 4y^3 - 2y$

$\dfrac{df}{dy} = 0$ when $y = \pm \dfrac{1}{\sqrt{2}}$ and $y = 0$

We therefore have six intervals of interest.



$\dfrac{dy}{dt} = f(y)$ $\qquad \dfrac{d^2y}{dt^2} = \left(\dfrac{df}{dy}\right) \times f(y)$

$f(y) = 0$ at $y = 0, -1, +1$

$\frac{df}{dy} = 0$ at $y = \frac{-1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}}$

Interval	$f(y) = \frac{dy}{dt}$	$\frac{df}{dy}$	$\frac{d^2y}{dt^2} = f(y)\frac{df}{dy}$
$(-\infty, -1]$	>0	<0	<0
$[-1, -\frac{1}{\sqrt{2}}]$	<0	<0	>0
$[-\frac{1}{\sqrt{2}}, 0]$	<0	>0	<0
$[0, \frac{1}{\sqrt{2}}]$	<0	<0	>0
$[\frac{1}{\sqrt{2}}, 1]$	<0	>0	<0
$[1, \infty)$	>0	>0	>0