Solutions to Assignment-1

Problem 1: Solve the following algebraic equations using perturbation techniques you learnt in class for $\varepsilon \ll 1$. Obtain all solutions correct upto 2nd order.

1. $x^2 + x + 6\varepsilon = 0.$
2. $x^3 - \varepsilon x - 1 = 0.$
3. $\varepsilon x + 3y = 10, \quad 4x + 2y = 7.$ Obtain x and y simultaneously correct to $O(\varepsilon^2)$.
4. $x^3 - x + \varepsilon = 0.$

A (1) $x^2 + x + 6\epsilon = 0$

Let $x = x_0 + \epsilon x_1 + \epsilon^2 x_2 + \cdots$

Substituting the asymptotic expression into the algebraic eqn; we get

$$(x_0 + \epsilon x_1 + \epsilon^2 x_2 + \cdots)^2 + (x_0 + \epsilon x_1 + \epsilon^2 x_2 + \cdots) + 6\epsilon = 0$$

$$\Rightarrow (x_0^2 + x_0) + \epsilon(2x_0 x_1 + x_1 + 6) + \epsilon^2(x_1^2 + 2x_0 x_2 + x_2) + O(\epsilon^3) = 0$$

$\underline{O(1)}$: $x_0^2 + x_0 = 0 \Rightarrow x_0 = 0$ or $x_0 = -1$

$\underline{O(\epsilon)}$: $(2x_0 + 1)x_1 + 6 = 0 \Rightarrow x_1 = -\dfrac{6}{2x_0+1}$ $\Big\}$ $x_1 = -6$ with $x_0 = 0$; $x_1 = 6$ with $x_0 = -1$

$\underline{O(\epsilon^2)}$: $x_2(2x_0 + 1) + x_1^2 = 0 \Rightarrow x_2 = \dfrac{-x_1^2}{2x_0+1}$

With $x_0 = 0$ & $x_1 = -6$, $x_2 = \dfrac{-36}{1} = -36$

With $x_0 = -1$ & $x_1 = 6$, $x_2 = \dfrac{-36}{-1} = 36$

$$\therefore \quad x^{(1)} = 0 - 6\epsilon - 36\epsilon^2 + \cdots$$
$$x^{(2)} = -1 + 6\epsilon + 36\epsilon^2 + \cdots$$

(2) $x^3 - \epsilon x - 1 = 0$

Let $x = x_0 + \epsilon x_1 + \epsilon^2 x_2 + \cdots$

Substituting the asymptotic expression into the algebraic eqn; we get

$$(x_0 + \epsilon x_1 + \epsilon^2 x_2 + \cdots)^3 - \epsilon(x_0 + \epsilon x_1 + \epsilon^2 x_2 + \cdots) - 1 = 0$$

$$\Rightarrow (x_0^3 - 1) + \epsilon(3x_0^2 x_1 - x_0) + \epsilon^2(3x_0^2 x_2 + 3x_0 x_1^2 - x_1) + O(\epsilon^3) = 0$$

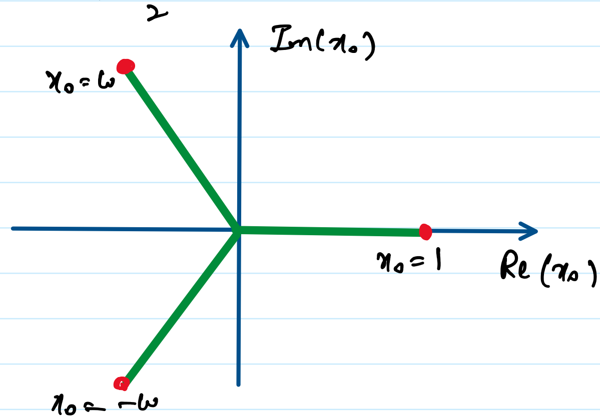
At $\underline{O(1)}$: $x_0^3 - 1 = 0 \longrightarrow x_0 = 1, \omega, \omega^2$

At $O(1)$: $x_0^3 - 1 = 0 \longrightarrow x_0 = 1, \omega, \omega^2$

where 1, $\omega = \frac{-1+i\sqrt{3}}{2}$, $\omega^2 = \frac{-1-i\sqrt{3}}{2}$ are the cube roots of unity.

Note that $1 \times \omega \times \omega^2 = 1$

& $1 + \omega + \omega^2 = 0$



At $O(\epsilon)$: $3x_0^2 x_1 - x_0 = 0 \Rightarrow x_1 = \frac{x_0}{3x_0^2} = \frac{1}{3x_0}$

$\therefore x_1 = \frac{1}{3} \quad (x_0 = 1)$

$= \frac{1}{3\omega} = \frac{\omega^3}{3\omega} = \frac{\omega^2}{3}$ for $x_0 = \omega$

$= \frac{1}{3\omega^2} = \frac{\omega}{3}$ for $x_0 = \omega^2$

At $O(\epsilon^2)$: $3x_0^2 x_2 + 3x_0 x_1^2 - x_1 = 0$

$\Rightarrow 3x_0^2 x_2 = x_1 - 3x_0 x_1^2 \Rightarrow x_2 = \frac{x_1 - 3x_0 x_1^2}{3x_0^2}$

$\therefore x_2 = \frac{1 - 3\times 1 \times \frac{1}{3}}{3 \times 1} = \frac{1-1}{3} = 0$ for $x_0 = 1$ & $x_1 = \frac{1}{3}$

$x_2 = \frac{\frac{\omega^2}{3} - 3\cdot\omega\cdot\frac{\omega^4}{9}}{3\cdot\omega^2} = \frac{\frac{\omega^2}{3} - \frac{\omega^2\cdot\omega^3}{3}}{3\omega^2} = \frac{\frac{\omega^2}{3} - \frac{\omega^2}{3}}{3\omega^2} = 0$

with $x_0 = \omega$ & $x_1 = \frac{\omega^2}{3}$

$x_2 = \frac{\frac{\omega}{3} - 3\cdot\omega^2\cdot\frac{\omega^2}{9}}{3\cdot\omega^2} = \frac{\frac{\omega}{3} - \frac{\omega\cdot\omega^3}{3}}{3\omega^2} = \frac{\frac{\omega}{3} - \frac{\omega}{3}}{3\omega^2} = 0$

with $x_0 = \omega^2$ & $\omega_1 = \frac{\omega}{3}$

Three roots:-

$x^{(1)} = 1 + \frac{\epsilon}{3} + O(\epsilon^3)$

$$x^{(1)} = 1 + \frac{\epsilon}{3} + O(\epsilon^3)$$

$$x^{(2)} = \omega + \frac{\omega^2}{3}\epsilon + O(\epsilon^3)$$

$$x^{(3)} = \omega^2 + \frac{\omega}{3}\epsilon + O(\epsilon^3)$$

(2) $\epsilon x + 3y = 10$; $4x + 2y = 7$

$$\begin{bmatrix} \epsilon & 3 \\ 4 & 2 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 10 \\ 7 \end{bmatrix}$$

$$A \quad X = B$$

$$\Rightarrow X = A^{-1}B$$

Let $x = x_0 + \epsilon x_1 + \epsilon^2 x_2 + \cdots$

$y = y_0 + \epsilon y_1 + \epsilon^2 y_2 + \cdots$

$$\epsilon(x_0 + \epsilon x_1 + \epsilon^2 x_2 + \cdots) + 3(y_0 + \epsilon y_1 + \epsilon^2 y_2 + \cdots) = 10$$

& $$4(x_0 + \epsilon x_1 + \epsilon^2 x_2 + \cdots) + 2(y_0 + \epsilon y_1 + \epsilon^2 y_2 + \cdots) = 7$$

$O(1)$: $\boxed{3y_0 = 10}$; $\boxed{4x_0 + 2y_0 = 7}$ $\Rightarrow y_0 = \frac{10}{3}$; $x_0 = \frac{7 - 2y_0}{4}$

$$\Rightarrow x_0 = \frac{7 - 2\cdot\frac{10}{3}}{4} = \frac{1}{12}$$

$O(\epsilon)$: $x_0 + 3y_1 = 0$; $4x_1 + 2y_1 = 0$ $\Rightarrow y_1 = \frac{-x_0}{3} = \frac{-1}{36}$

& $x_1 = \frac{-2y_1}{4} = \frac{1}{72}$

(3) $x^3 - x + \epsilon = 0$

Let $x = x_0 + \epsilon x_1 + \epsilon^2 x_2 + \cdots$

Substituting the asymptotic expression into the algebraic eqn; we get

$$(x_0 + \epsilon x_1 + \epsilon^2 x_2 + \cdots)^3 - (x_0 + \epsilon x_1 + \epsilon^2 x_2 + \cdots) + \epsilon = 0$$

$O(1)$: $x_0^3 - x_0 = 0$ $\Rightarrow$ $x_0 = 0, -1, +1$

$O(\epsilon)$: $3x_0^2 x_1 - x_1 + 1 = 0 \Rightarrow (3x_0^2 - 1)x_1 = -1 \Rightarrow x_1 = \frac{1}{1 - 3x_0^2}$

$\therefore x_1 = 1$ with $x_0 = 0$

$= \frac{1}{-2}$ with $x_0 = -1$

$= \frac{-1}{}$ with $x_0 = +1$

$$-2$$

$$= \frac{-1}{2} \quad \text{with} \quad x_0 = +1$$

$\underline{O(\epsilon^2)}$: $\quad 3x_0 x_2 - x_2 + 3x_0 x_1^2 = 0 \Rightarrow x_2(3x_0 - 1) = -3x_0 x_1^2$

$$\Rightarrow x_2 = \frac{3x_0 x_1^2}{1-3x_0}$$

$$\therefore x_2 = 0 \quad \text{for} \quad x_0 = 0, \ x_1 = 1$$

$$x_2 = \frac{3\times(-1)\times 1/4}{1-3\times(-1)} = \frac{-3/4}{1+3} = \frac{-3}{16} \quad \text{with} \quad x_0 = -1, \ x_1 = -1/2$$

$$x_2 = \frac{3\times 1\times 1/4}{1-3\times 1} = \frac{3/4}{-2} = \frac{-3}{8} \quad \text{with} \quad x_0 = 1, \ x_1 = -1/2$$

<u>Three roots</u>:-

$$x^{(1)} = \epsilon + O(\epsilon^3)$$

$$x^{(2)} = -1 - \frac{\epsilon}{2} - \frac{3}{16}\epsilon^2 + O(\epsilon^3)$$

$$x^{(3)} = 1 - \frac{\epsilon}{2} - \frac{3}{8}\epsilon^2 + O(\epsilon^3)$$

Problem 2: First obtain the exact solution of

$$x^2 - \pi x + 2 = 0$$

Use a calculator and write down the exact solution upto six decimal places. Now pretend that $\pi = 3+\varepsilon$. Obtain the perturbation solution of

$$x^2 - (3+\varepsilon)x + 2 = 0$$

up to 3rd order. It is sufficient to obtain the solution of the smaller root. Finally, set the value of $\varepsilon = \pi - 3$ to obtain the approximate solution. How does the obtained approximate solution compare with the exact solution, i.e. what is the exact value of the error?

(A) <u>Exact Solution</u>:- $\quad x = \dfrac{\pi \pm \sqrt{\pi^2 - 4\times 2}}{2} = \dfrac{\pi}{2} \pm \sqrt{\dfrac{\pi^2}{4} - 2}$

$$= 2.254464\ldots \quad \text{and} \quad \underbrace{0.887129\ldots}_{\text{Smaller root}}$$

<u>Approximate Solution</u>:-

Rewriting $\pi = 3 + \epsilon$, the quadratic equation becomes

$$x^2 - (3+\epsilon)x + 2 = 0$$

Let $x = x_0 + \epsilon x_1 + \epsilon^2 x_2 + \epsilon^3 x_3 + \cdots$

Substituting, we have

$$\left(x_0^2 + 2\epsilon x_0 x_1 + \epsilon^2 x_1 + \epsilon^2 \cdot 2x_0 x_2 + \epsilon^3 \cdot 2x_0 x_3 + \epsilon^3 \cdot 2x_1 x_2 + \cdots \right)$$
$$- (3+\epsilon)\left(x_0 + \epsilon x_1 + \epsilon^2 x_2 + \epsilon^3 x_3 + \cdots \right) + 2 = 0$$

$\underline{\underline{O(\epsilon^0)}}$: $\quad x_0^2 - 3x_0 + 2 = 0 \quad \Rightarrow \quad \underbrace{x_0 = 1}_{\text{smaller root}} \quad$ or $\quad x_0 = 2$

$\underline{\underline{O(\epsilon^1)}}$: $\quad 2x_0 x_1 - 3x_1 - x_0 = 0 \quad \Rightarrow \quad x_1 = \dfrac{x_0}{2x_0 - 3} = -1 \quad$ for $x_0 = 1$

$\underline{\underline{O(\epsilon^2)}}$: $\quad 2x_0 x_2 + x_1^2 - 3x_2 - x_1 = 0 \quad \Rightarrow \quad x_2 = \dfrac{x_1 - x_1^2}{2x_0 - 3} = 2 \quad$ for $x_0 = 1$, $x_1 = -1$

$\underline{\underline{O(\epsilon^3)}}$: $\quad 2x_0 x_3 + 2x_1 x_2 - 3x_3 - x_2 = 0 \quad \Rightarrow \quad x_3 = \dfrac{x_2(1 - 2x_1)}{2x_0 - 3}$

$\Rightarrow \quad x_3 = -6 \quad$ for $x_0 = 1$; $x_1 = -1$; $x_2 = 2$

$\therefore$ The smaller root is

$$x^{(1)} = 1 - \epsilon + 2\epsilon^2 - 6\epsilon^3 + O(\epsilon^4)$$

With $\epsilon = \pi - 3 = 3.14159625 - 3 = 0.14159625$, we get

$$x^{(1)} \approx 0.881472$$

$$\text{Error} = x^{(1)}_{\text{exact}} - x^{(1)}_{\text{approx}}$$
$$= 0.887129 - 0.881472$$
$$= 0.005657 \approx 5.65 \times 10^{-3}$$

Problem 3 Solve the following non-algebraic equation using regular perturbation technique for $\varepsilon \ll 1$:

$$x^2 - 1 = \varepsilon e^x.$$

Here is some help. If we set $\varepsilon = 0$, the leading order solutions are $x = -1$ and $x = +1$. Using regular perturbation technique, obtain the approximate solution of the first solution, i.e., when $x_0 = -1$. You will have to substitute the perturbation expansion for x in $e^x = e^{x_0 + \varepsilon x_1 + \varepsilon x^2 + \ldots}$ and simplify into a polynomial expression. Obtain the solution upto 2nd order.

(A) If $\epsilon = 0$, we get $x^2 - 1 = 0 \Rightarrow x = -1$ & $x = +1$

polynomial expression. Obtain the solution upto 2nd order.

(A) If $\epsilon = 0$, we get $x^2 - 1 = 0 \Rightarrow x = -1$ & $x = +1$

Let us obtain regular perturbation expansion as follows:-

$$x = x_0 + \epsilon x_1 + \epsilon^2 x_2 + \cdots$$

$$\Rightarrow (x_0 + \epsilon x_1 + \epsilon^2 x_2 + \cdots)^2 - 1 = \epsilon e^{x_0 + \epsilon x_1 + \epsilon^2 x_2 + \cdots}$$

Let us first simplify $e^{\epsilon x}$

$$e^{x_0 + \epsilon x_1 + \epsilon^2 x_2 + \cdots} = e^{x_0} \cdot e^{\epsilon x_1 + \epsilon^2 x_2 + \cdots}$$

$$= e^{x_0}\left[1 + (\epsilon x_1 + \epsilon^2 x_2 + \cdots) + \frac{(\epsilon x_1 + \epsilon^2 x_2 + \cdots)^2}{2!} + \cdots\right]$$

$$= e^{x_0}\left[1 + \epsilon x_1 + \epsilon^2 x_2 + \frac{1}{2}\epsilon^2 x_1^2 + O(\epsilon^3)\right]$$

$$= e^{x_0}\left[1 + \epsilon x_1 + \epsilon^2\left(x_2 + \frac{1}{2}x_1^2\right) + O(\epsilon^3)\right]$$

$$\therefore \epsilon e^{x} = \epsilon e^{x_0} + \epsilon^2 e^{x_0} \cdot x_1 + O(\epsilon^3)$$

Putting everything together, we have

$$(x_0^2 + 2\epsilon x_0 x_1 + 2\epsilon^2 x_0 x_2 + \epsilon^2 x_1^2 + \cdots) - 1 = \epsilon e^{x_0} + \epsilon^2 e^{x_0} \cdot x_1 + O(\epsilon^3)$$

$\underline{\underline{O(1)}}$: $x_0^2 - 1 = 0 \Rightarrow x_0 = 1, \quad x_0 = -1$

We will look for a solution with $x_0 = -1$

$\underline{\underline{O(\epsilon)}}$: $2x_0 x_1 = e^{x_0} \Rightarrow x_1 = \dfrac{e^{x_0}}{2x_0} = \dfrac{e^{-1}}{2\times(-1)} = \dfrac{-1}{2e}$

$\underline{\underline{O(\epsilon^2)}}$: $2x_0 x_2 + x_1^2 = e^{x_0} \cdot x_1$

$$\Rightarrow 2\times(-1)\times x_2 + \left(\frac{-1}{2e}\right)^2 = e^{-1}\times\left(\frac{-1}{2e}\right)$$

$$\Rightarrow -2x_2 + \frac{1}{4e^2} = \frac{-1}{2e^2} \Rightarrow -2x_2 = \frac{-1}{2e^2} - \frac{1}{4e^2} = \frac{-3}{4e^2}$$

$$\therefore x_2 = \frac{3}{8e^2}$$

$$\therefore \boxed{x^{(1)} = -1 - \frac{1}{2e}\epsilon + \frac{3}{8e^2}\epsilon^2 + O(\epsilon^3)}$$

Problem 4 Solve the following algebraic equations using perturbation techniques you learnt in class for $\varepsilon \ll 1$. Obtain all solutions correct upto 1st order. Note that some of the below equations may have both regular and singular roots. Obtain all the roots.

1. $\varepsilon x^2 + x - 1 = 0$.
2. $\varepsilon x^3 - x + 1 = 0$.
3. $\varepsilon x^3 + x^2 - 2x + 1 = 0$.
4. $(1-\varepsilon)x^2 - 2x + 1 = 0$. (Hint: You will have a non-integral power for ε. You will get a repeated root for leading term which will lead to difficulties).

(A) 1 $\quad \epsilon x^2 + x - 1 = 0$

With $\epsilon = 0$, we get $x = 1 \Rightarrow$ Regular root.

$\therefore$ Let $x = x_0 + \epsilon x_1 + \epsilon^2 x_2 + \cdots$

$\therefore \epsilon(x_0 + \epsilon x_1 + \cdots)^2 + (x_0 + \epsilon x_1 + \cdots) - 1 = 0$

$\underline{O(1)}$: $x_0 - 1 = 0 \Rightarrow x_0 = 1$

$\underline{O(\epsilon)}$: $x_0 + x_1 = 0 \Rightarrow x_1 = -x_0 = -1$

$$\therefore x_{regular} = 1 - \epsilon + O(\epsilon^2)$$

Singular root:- Let $x = \delta X$ with $X \sim Ord(1)$

Substituting, $\epsilon\delta^2 X^2 + \delta X - 1 = 0$

Using method of dominant balance:

(i) $\epsilon\delta^2 \sim 1 \Rightarrow \delta \sim \frac{1}{\sqrt{\epsilon}}$

This gives: $\epsilon \cdot \frac{1}{\epsilon}X^2 + \frac{1}{\sqrt{\epsilon}}X - 1 = 0$

$$\Rightarrow \underbrace{X^2}_{O(1)} + \underbrace{\frac{1}{\sqrt{\epsilon}}}_{\substack{O\left(\frac{1}{\sqrt{\epsilon}}\right)\\ \text{large}}} X \underbrace{- 1}_{O(1)} = 0$$

Equation is unbalanced.

(ii) $\epsilon\delta^2 \sim \delta \Rightarrow \epsilon\delta \sim 1 \Rightarrow \delta = \frac{1}{\epsilon}$

(ii) $\epsilon\delta \sim \delta \Rightarrow \epsilon\delta \sim 1 \Rightarrow \delta \sim \frac{1}{\epsilon}$

This gives: $\epsilon \cdot \frac{1}{\epsilon^2} X^2 + \frac{1}{\epsilon} X - 1 = 0$

$$\left.\Rightarrow \underset{O(1)}{X^2} + \underset{O(1)}{X} - \underset{\substack{O(\epsilon)\\ \text{small}}}{\epsilon} = 0 \right\}$$ Equation is balanced.

$\therefore$ We take $\delta = \frac{1}{\epsilon}$ to obtain

$$X^2 + X - \epsilon = 0$$

$$X = X_0 + \epsilon X_1 + \epsilon^2 X_2 + \cdots$$

$$(X_0 + \epsilon X_1 + \epsilon^2 X_2 + \cdots)^2 + (X_0 + \epsilon X_1 + \epsilon^2 X_2 + \cdots) - \epsilon = 0$$

$\underline{\underline{O(1)}}$: $X_0^2 + X_0 = 0 \Rightarrow X_0(X_0 + 1) = 0 \Rightarrow X_0 = 0$ or $X_0 = -1$

$\underline{\underline{O(\epsilon)}}$: $2X_0 X_1 + X_1 - 1 = 0 \Rightarrow X_1(2X_0 + 1) = 1 \Rightarrow X_1 = \frac{1}{2X_0 + 1}$

$\therefore X_1 = 1$ if $X_0 = 0$

$X_1 = \frac{1}{-1} = -1$ if $X_0 = -1$

$\underline{\underline{O(\epsilon^2)}}$: $X_1^2 + 2X_0 X_2 + X_2 = 0 \Rightarrow X_2(2X_0 + 1) = -X_1^2$

$\therefore X_2 = \frac{-X_1^2}{1 + 2X_0}$

$\therefore X_2 = \frac{-1}{1 + 2\times 0} = -1$ if $X_0 = 0$ & $X_1 = 1$

& $X_2 = \frac{-1}{1-2} = 1$ if $X_0 = -1$ & $X_1 = -1$

$\therefore X^{(1)} = 0 + \epsilon \times 1 - 1 \times \epsilon^2 + O(\epsilon^3)$

$X^{(2)} = -1 + \epsilon(-1) + 1 \times \epsilon^2 + O(\epsilon^3)$

$\therefore x^{(1)} = \frac{X^{(1)}}{\epsilon} = 1 - \epsilon + O(\epsilon^2)$: Regular root

$x^{(2)} = \frac{X^{(2)}}{\epsilon} = \frac{-1}{\epsilon} - 1 + \epsilon + O(\epsilon^2)$: Singular root

(2) $\epsilon x^3 - x + 1 = 0$

If $\epsilon = 0$, we get $-x+1=0 \Rightarrow x=1$: This gives us the regular root.

Take $x = x_0 + \epsilon x_1 + \epsilon^2 x_2 + \cdots$

$\therefore \epsilon(x_0 + \epsilon x_1 + \cdots)^3 - (x_0 + \epsilon x_1 + \cdots) + 1 = 0$

$O(1)$: $-x_0 + 1 = 0 \Rightarrow x_0 = 1$

$O(\epsilon)$: $x_0^3 - x_1 = 0 \Rightarrow x_1 = x_0^3 = 1$

$\therefore x_{\text{regular}} = 1 + \epsilon + O(\epsilon^2)$

Singular roots:- Let $x = \delta X$

$\Rightarrow \epsilon \delta^3 X^3 - \delta X + 1 = 0$

Dominant balance:- (i) $\epsilon \delta^3 = 1 \Rightarrow \delta^3 = \frac{1}{\epsilon} \Rightarrow \delta = \frac{1}{\epsilon^{1/3}}$

$\epsilon \cdot \frac{1}{\epsilon} X^3 - \frac{1}{\epsilon^{1/3}} X + 1 = 0$

$X^3 - \underbrace{\frac{1}{\epsilon^{1/3}}}_{\text{large}} X + 1 = 0$

$\longrightarrow$ Eqn. is unbalanced.

(ii) $\delta = 1$: This gives us the regular root.

(iii) $\epsilon \delta^3 = \delta \Rightarrow \epsilon \delta^2 = 1 \Rightarrow \delta = \frac{1}{\sqrt{\epsilon}}$

$\therefore \epsilon \cdot \frac{1}{\epsilon^{3/2}} X^3 - \frac{1}{\epsilon^{1/2}} X + 1 = 0$

$\Rightarrow \frac{X^3}{\epsilon^{1/2}} - \frac{X}{\epsilon^{1/2}} + 1 = 0$

$\Rightarrow X^3 - X + \epsilon^{1/2} = 0$

$O(1) \quad O(1) \quad \underbrace{O(\epsilon^{1/2})}_{\text{small}} \longrightarrow$ Equation is balanced.

$\therefore \delta = \frac{1}{\epsilon^{1/2}}$

$X^3 - X + \epsilon^{1/2} = 0$

Let $X = X_0 + \epsilon^{1/2} X_1 + \epsilon X_2 + \cdots$

$(X_0 + \epsilon^{1/2} X_1 + \epsilon X_2 + \cdots)^3 - (X_0 + \epsilon^{1/2} X_1 + \epsilon X_2 + \cdots) + \epsilon^{1/2} = 0$

$$(X_0 + \epsilon^{1/2}X_1 + \epsilon X_2 + \cdots)^3 - (X_0 + \epsilon^{1/2}X_1 + \epsilon X_2 + \cdots) + \epsilon^{1/2} = 0$$

$\underline{O(1)}$: $X_0^3 - X_0 = 0 \Rightarrow X_0(X_0^2 - 1) = 0 \Rightarrow \underbrace{X_0 = 0}_{\text{This gives us the regular root}}, \; X_0 = \pm 1$

$\underline{O(\epsilon^{1/2})}$ $3X_0^2 X_1 - X_1 + 1 = 0$

$\Rightarrow X_1(3X_0^2 - 1) = -1 \Rightarrow X_1 = \dfrac{1}{1 - 3X_0^2}$

with $X_0 = 0$; $X_1 = 1$

with $X_0 = -1$; $X_1 = \dfrac{1}{1-3} = \dfrac{-1}{2}$

with $X_0 = 1$; $X_1 = \dfrac{-1}{2}$

$\underline{O(\epsilon)}$: $3X_0^2 X_2 + 3X_0 X_1^2 - X_2 = 0 \Rightarrow X_2(3X_0^2 - 1) = -3X_0 X_1^2$

$\therefore X_2 = \dfrac{3X_0 X_1^2}{1 - 3X_0^2}$

with $X_0 = 0$, $X_1 = 1$; $X_2 = 0$

with $X_0 = -1$, $X_1 = \frac{-1}{2}$, $X_2 = \dfrac{3 \times (-1) \times 1/4}{1 - 3 \times 1} = \dfrac{-3/4}{-2} = \dfrac{3}{8}$

with $X_0 = 1$, $X_1 = \frac{-1}{2}$, $X_2 = \dfrac{3 \times 1 \times 1/4}{1 - 3 \times 1} = \dfrac{3}{8}$

$\therefore X^{(1)} = \epsilon^{1/2} + O(\epsilon^{3/2}) + \cdots \Rightarrow x^{(1)} = \dfrac{X^{(1)}}{\epsilon^{1/2}} = 1 + O(\epsilon)$

$X^{(2)} = -1 - \frac{1}{2}\epsilon^{1/2} + \frac{3}{8}\epsilon + O(\epsilon^{3/2}) \Rightarrow x^{(2)} = \dfrac{-1}{\epsilon^{1/2}} - \frac{1}{2} + \frac{3}{8}\epsilon^{1/2} + O(\epsilon)$

$X^{(3)} = 1 - \frac{1}{2}\epsilon^{1/2} + \frac{3}{8}\epsilon + O(\epsilon^{3/2}) \Rightarrow x^{(3)} = \dfrac{1}{\epsilon^{1/2}} - \frac{1}{2} + \frac{3}{8}\epsilon^{1/2} + O(\epsilon)$

All $\underline{\text{three roots}}$:-

$$\boxed{\begin{aligned} x^{(1)}_{\text{regular}} &= 1 + \epsilon + O(\epsilon^2) \\ x^{(2)}_{\text{singular}} &= \frac{-1}{\epsilon^{1/2}} - \frac{1}{2} + \frac{3}{8}\epsilon^{1/2} + O(\epsilon) \\ x^{(3)}_{\text{singular}} &= \frac{1}{\epsilon^{1/2}} - \frac{1}{2} + \frac{3}{8}\epsilon^{1/2} + O(\epsilon) \end{aligned}}$$

③ $\epsilon x^3 + x^2 - 2x + 1 = 0$

Let $x = x_0 + \epsilon x_1 + \epsilon^2 x_2 + \cdots$

$\epsilon(x_0 + \epsilon x_1 + \cdots)^3 + (x_0 + \epsilon x_1 + \cdots)^2 - 2(x_0 + \epsilon x_1 + \cdots) + 1 = 0$

$\underline{O(1)}$: $x_0^2 - 2x_0 + 1 = 0 \Rightarrow x_0 = \underbrace{1, 1}_{\text{double root}}$

$\underline{O(\epsilon)}$: $x_0^3 + 2x_0x_1 - 2x_1 + 1 = 0 \Rightarrow 2x_1 - 2x_1 = -1$

This cannot be satisfied for any value of x_1.

To obtain the regular root with correct expansion, let us expand the solution about $x_0 = 1$:

Let $x = 1 + \delta X$ where $X \sim Ord(1)$
& $\delta << 1$

Substituting, we get

$\epsilon\left[1 + \delta^3 X^3 + 3\delta^2 X + 3\delta X^2\right] + \left[1 + \delta^2 X^2 + 2\delta X\right] - 2(1 + \delta X) + 1 = 0$

$\Rightarrow \epsilon + \epsilon\delta^3 X^3 + 3\epsilon\delta^2 X + 3\epsilon\delta X^2 + \delta^2 X^2 = 0$

Since ϵ & δ are both small, there are two possibilities:-

(i) $\epsilon \sim \epsilon\delta \Rightarrow \delta \sim 1$ which violates $\delta << 1$

(ii) $\epsilon \sim \delta^2 \Rightarrow \delta = \epsilon^{1/2}$

Substituting $\delta = \epsilon^{1/2}$, we get

$\epsilon(1 + X^2) + 3X\epsilon^2 + 3X^2\epsilon^{3/2} + \epsilon^{5/2}X^3 = 0$

Again using $X = X_0 + \epsilon^{1/2} X_1 + \epsilon X_2 + \cdots$

$\underline{O(\epsilon)}$: $X_0^2 + 1 = 0 \Rightarrow X_0 = \pm i$

With $\delta = \epsilon^{1/2}$ & $X_0 = \pm i$, we now have

$\boxed{x = 1 \pm \epsilon^{1/2} i + O(\epsilon)}$: Regular roots

$\underline{\text{Singular}}$ $\underline{\text{root}}$:- $\epsilon x^3 + x^2 - 2x + 1 = 0$

We can balance two terms at a time and determine if the equation is balanced or not :-

(i) Let $\epsilon x^3 \sim x^2 \Rightarrow \epsilon x \sim 1 \Rightarrow x \sim \frac{1}{\epsilon}$

With $x \sim \frac{1}{\epsilon}$, we can now compare the strength of various terms :-

$$\underbrace{O\left(\frac{1}{\epsilon^2}\right)}_{\text{very large}} + \underbrace{O\left(\frac{1}{\epsilon^2}\right)}_{\text{very large}} - 2\underbrace{O\left(\frac{1}{\epsilon}\right)}_{\text{large}} + \underbrace{1}_{O(1)} = 0$$

These terms balance each other

$\therefore$ The equation can be balanced when $x \sim \frac{1}{\epsilon}$

(ii) Let $\epsilon x^3 \sim x \Rightarrow \epsilon x^2 \sim 1 \Rightarrow x \sim \frac{1}{\epsilon^{1/2}}$

Strength of various terms :-

$$\underbrace{O\left(\frac{1}{\epsilon^{1/2}}\right)}_{\text{large}} + \underbrace{O\left(\frac{1}{\epsilon}\right)}_{\text{very large}} + \underbrace{O\left(\frac{1}{\epsilon^{1/2}}\right)}_{\text{large}} + 1 = 0$$

This term cannot be balanced.

(iii) $\epsilon x^3 \sim 1 \Rightarrow x \sim \frac{1}{\epsilon^{1/3}}$

Strength of various terms :-

$$O(1) + \underbrace{O\left(\frac{1}{\epsilon^{2/3}}\right)}_{\text{very large}} + \underbrace{O\left(\frac{1}{\epsilon^{1/3}}\right)}_{\text{large}} + 1 = 0$$

The equation cannot be balanced again.

We therefore consider $x \sim O\left(\frac{1}{\epsilon}\right)$ based on option (i).
Let us expand x as follows :-

$$x = \frac{1}{\epsilon}x_{-1} + x_0 + \epsilon x_1 + \epsilon^2 x_2 + \cdots$$

Substituting :-

$$\epsilon\left[\frac{1}{\epsilon}x_{-1} + x_0 + \cdots\right]^3 + \left[\frac{1}{\epsilon}x_{-1} + x_0 + \epsilon x_1 + \cdots\right]^2$$

$$-2\left[\frac{1}{\epsilon}x_{-1} + x_0 + \epsilon x_1 + \cdots\right] + 1 = 0$$

$$-2\left[\frac{1}{\epsilon}x_{-1} + x_0 + \epsilon x_1 + \cdots\right] + 1 = 0$$

The largest term is $O\left(\frac{1}{\epsilon^2}\right)$. Hence we start comparing terms from this order.

$O\left(\frac{1}{\epsilon^2}\right)$: $\quad x_{-1}^3 + x_{-1}^2 = 0 \quad \Rightarrow \quad x_{-1}^2\,(x_{-1}+1) = 0$

$$\Rightarrow \quad \underbrace{x_{-1} = 0, 0}_{\text{This belongs to the regular root}} \quad \text{or} \quad x_{-1} = -1$$

Hence, we ignore for now.

<u>$O\left(\frac{1}{\epsilon}\right)$:</u> $\quad -2x_{-1} + 2x_0 x_{-1} + 3x_0 x_{-1}^2 = 0$

with $x_{-1} = -1$, we get $x_0 = -2$

<u>$O(1)$:</u> $\quad 1 - 2x_0 + x_0^2 + 3x_0^2 x_{-1} + 2x_1 x_{-1} + 3x_1 x_{-1}^2 = 0$

with $x_{-1} = -1$ & $x_0 = -2$, we get
$x_1 = 3$

We now have

$$x = -\frac{1}{\epsilon} - 2 + 3\epsilon + O(\epsilon^2)$$

<u>Summary:-</u> <u>Regular roots:-</u>
$$x^{(1)} = 1 + i\,\epsilon^{1/2} + O(\epsilon)$$
$$x^{(2)} = 1 - i\,\epsilon^{1/2} + O(\epsilon)$$

<u>Singular root:-</u>
$$x^{(3)} = -\frac{1}{\epsilon} - 2 + 3\epsilon + O(\epsilon^2)$$

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4 $\quad (1-\epsilon)x^2 - 2x + 1 = 0$

If we proceed with the usual expansion with ϵ, we get

$$x = x_0 + \epsilon x_1 + \epsilon^2 x_2 + \cdots$$

Substituting, we get

$$(x_0^2 - 2x_0 + 1) + \epsilon\,(2x_0x_1 - 2x_1 - x_0^2) + \epsilon^2\,(2x_0x_2 - 2x_2 + x_1^2 - 2x_0x_1) + O(\epsilon^3) = 0$$

<u>$O(1)$:</u> $\quad x_0^2 - 2x_0 + 1 = 0 \quad \Rightarrow \quad \underbrace{x_0 = 1, 1}_{\text{double root}}$

Usually double root means trouble. If we proceed further, we get:

$\underline{\underline{O(\epsilon)}}$: $2x_0 x_1 - 2x_1 - x_0^2 = 0$

$\Rightarrow 2x_1 - 2x_1 = 1$

This cannot be satisfied for any value of x_1.

This is not a singular perturbation expansion in the usual sense since the highest degree term does not vanish at leading order. Instead, at leading order, what we get is a double root.

To examine why the $O(\epsilon)$ expression fails, let us take a look at the exact solution:-

$$x_{exact} = \frac{1 \pm \epsilon^{1/2}}{1-\epsilon}$$

The exact solution suggests that the correct expansion power is $\epsilon^{1/2}$, not ϵ.

But notice that we have a double root and our leading solution, i.e; x_0, is OK. We, therefore, look for a correction from x_0 in the following form:-

Let $x = 1 + \delta X$ $\quad \underline{\underline{\delta << 1}}$

leading solution

$X \sim Ord(1)$ strictly

Substituting into the equation, we get

$$(1-\epsilon)(1+\delta X)^2 - 2(1+\delta X) + 1 = 0$$

$$\Rightarrow (1-\epsilon)[1+\delta^2 X^2 + 2\delta X] - 2 - 2\delta X + 1 = 0$$

$$\Rightarrow \cancel{1} + \delta^2 X^2 + \cancel{2\delta X} - \epsilon - \epsilon\delta^2 X^2 - 2\epsilon\delta X - \cancel{2} - \cancel{2\delta X} + \cancel{1} = 0$$

$$\Rightarrow \delta^2 X^2 - \epsilon - \underbrace{\epsilon\delta^2 X^2}_{\text{Very small}} - 2\epsilon\delta X = 0$$

There are two possibilities:-

(i) $\delta^2 \sim \epsilon\delta \Rightarrow \delta \sim \epsilon$

This is no different from what has already been done, i.e; when we noticed that an expansion in ϵ fails.

(ii) $\delta^2 \sim \epsilon \Rightarrow \delta \sim \epsilon^{1/2}$

This allows us to set up an expansion about $x_0 = 1$ as:

$$x = 1 + \epsilon^{1/2} X + \cdots$$

Substituting again & simplifying, we get

$$\epsilon X^2 - \epsilon - \epsilon^2 X^2 - 2\epsilon^{3/2} X = 0$$

<u>$O(\epsilon)$</u>: $X^2 - 1 = 0 \Rightarrow X = \pm 1$

The solution now becomes

$$x = 1 + \delta X + O(\epsilon)$$

$$= 1 + \epsilon^{1/2} (\pm 1) + O(\epsilon)$$

$$\Rightarrow \boxed{\begin{aligned} x^{(1)} &= 1 - \epsilon^{1/2} + O(\epsilon) \\ \& \quad x^{(2)} &= 1 + \epsilon^{1/2} + O(\epsilon) \end{aligned}}$$