Singular Perturbation Theory :-

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Ex: $\epsilon y'' + y' = e^{-x}$; $y(0) = 1$; $y(1) = 1$

Let $y(x) = y_0(x) + \epsilon y_1(x) + \epsilon^2 y_2(x) + \cdots$

Substituting :-

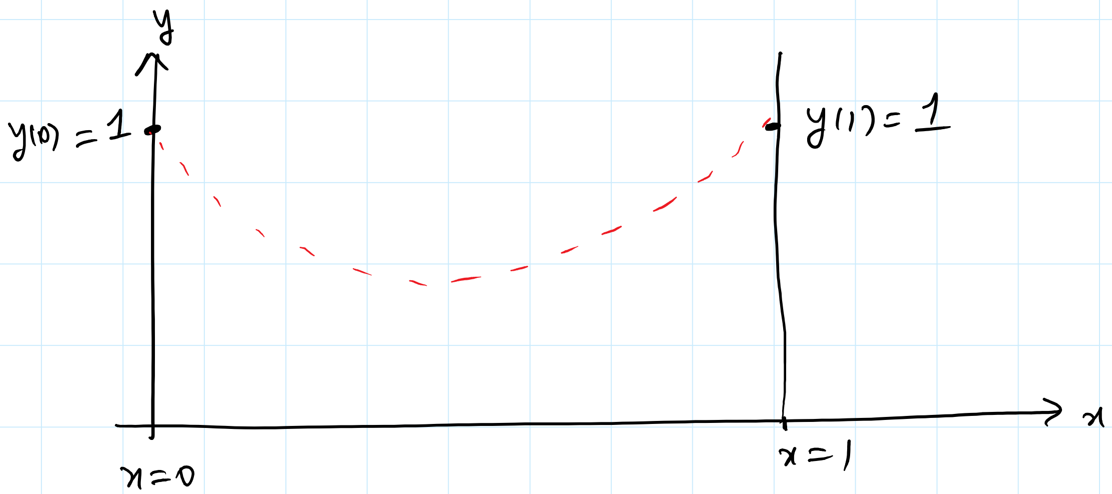
$$\epsilon\left[y_0'' + \epsilon y_1'' + \epsilon^2 y_2'' + \cdots\right] + \left[y_0' + \epsilon y_1' + \epsilon^2 y_2' + \cdots\right] = e^{-x}$$

with
$$y_0(0) + \epsilon y_1(0) + \epsilon^2 y_2(0) + \cdots = 1$$
$$y_0(1) + \epsilon y_1(1) + \epsilon^2 y_2(1) + \cdots = 1$$

O(1): $y_0' = e^{-x}$

$\Rightarrow y_0(x) = -e^{-x} + C_0$

$\Rightarrow \boxed{y_0(x) = C_0 - e^{-x}}$



To find C_0; we need to apply the boundary condition at $x = 0$ or $x = 1$.

Which of these is the correct choice?

If $y_0(0) = 1$ is the correct choice, then we get

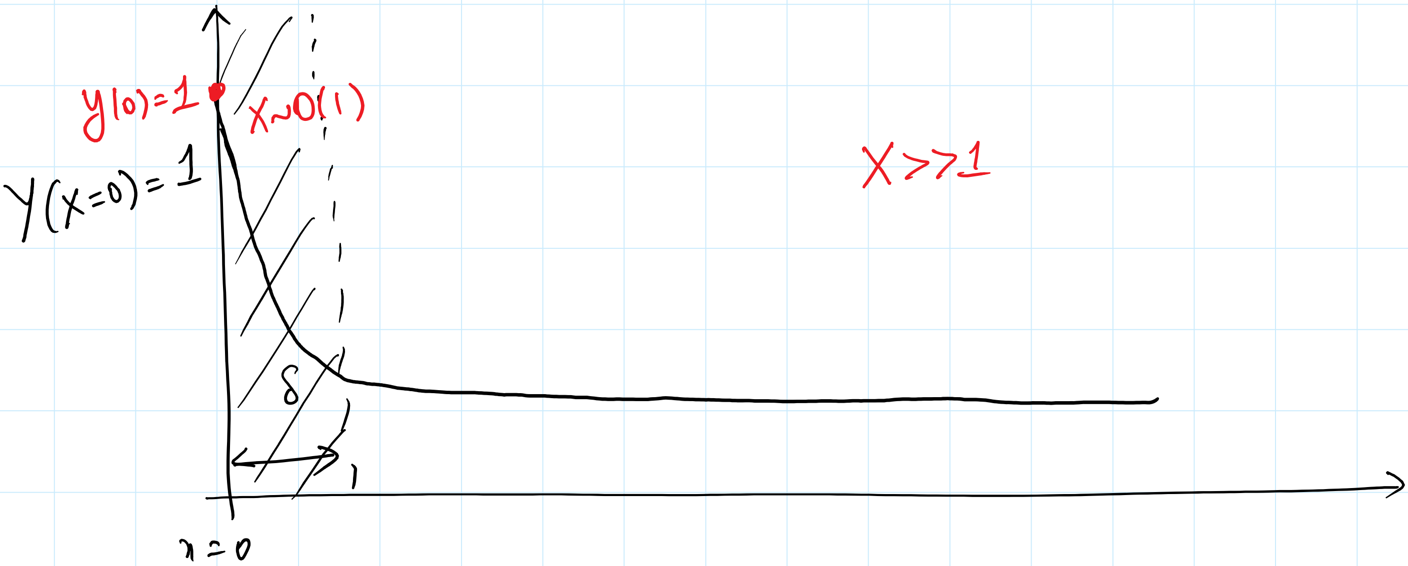
$1 = C_0 - e^0 \Rightarrow 1 = C_0 - 1 \Rightarrow C_0 = 2$

If $y_0(1) = 1$ is the correct choice, then we get

$1 = C_0 - e^{-1} \Rightarrow 1 = C_0 - \frac{1}{e} \Rightarrow C_0 = 1 + \frac{1}{e}$

To resolve this, let us probe the solution near each boundary :-

Let us focus near $x = 0$:-



Let $\boxed{x = \delta X}$ when $X \sim Ord(1)$ & $\delta << 1$

$\Rightarrow x << 1$

& $y(x) = Y(X)$

$\therefore \frac{dy}{dx} = \frac{dy}{dX} \cdot \frac{dX}{dx}$

$$\frac{dx}{} - \frac{dX}{dx}$$

$$\boxed{\frac{dy}{dx} = \frac{dY}{dX} \cdot \frac{1}{\delta}}$$

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d}{dX}\left(\frac{dy}{dx}\right) \cdot \frac{dX}{dx}$$

$$= \frac{d}{dX}\left(\frac{1}{\delta} \cdot \frac{dY}{dX}\right) \cdot \frac{1}{\delta} = \frac{1}{\delta^2} \cdot \frac{d^2Y}{dX^2}$$

Substituting, we get

$$\epsilon \cdot \frac{1}{\delta^2} \frac{d^2Y}{dX^2} + \frac{1}{\delta} \frac{dY}{dX} = e^{-\delta X}$$

$$\Rightarrow \quad \frac{\epsilon}{\delta} \cdot \frac{d^2Y}{dX^2} + \frac{dY}{dX} = \delta e^{-\delta X}$$

$$\frac{\epsilon}{\delta} \, O(1) + O(1) = \text{Very Small}$$

Natural choice for δ to balance the equation is

$$\boxed{\delta = \epsilon}$$

Equation becomes:- $$\frac{d^2Y}{dX^2} + \frac{dY}{dX} = \epsilon e^{-\epsilon X}$$

Let $Y(X) = Y_0(X) + \epsilon Y_1(X) + \epsilon^2 Y_2(X) + \cdots$

$O(1)$: $$\frac{d^2Y_0}{dX^2} + \frac{dY_0}{dX} = 0$$

with $Y_0(x=0) = 1$

$$\underbrace{\frac{dY_0}{dx} + Y_0}= c_1$$

Multiply by e^x: $e^x \dfrac{dY_0}{dx} + Y_0 = c_1 e^x$

$$\Rightarrow \frac{d}{dx}\left(e^x Y_0\right) = c_1 e^x$$

$$Y_0 e^x = c_1 e^x + c_2$$

$$\Rightarrow Y_0(x) = c_1 + c_2 e^{-x}$$

With $Y_0(0) = 1$, we get $1 = c_1 + c_2$

$$\Rightarrow c_2 = 1 - c_1$$

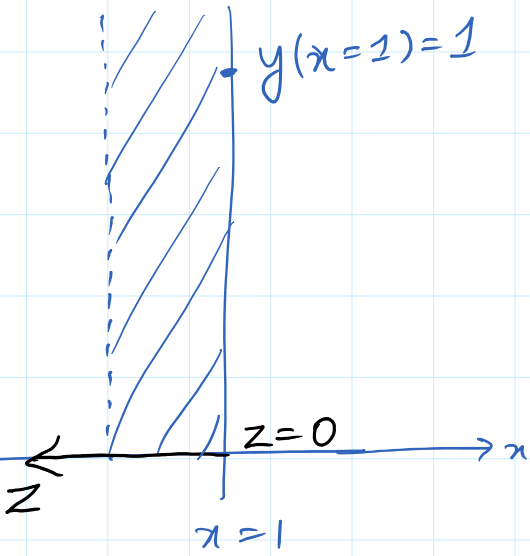
$$\therefore \; Y_0(x) = c_1 + (1-c_1)e^{-x}$$

$$\boxed{Y_0(x) = e^{-x} + c_1(1 - e^{-x})}$$

Let us focus on the region near $x = 1$:-

$x = 1 - \delta Z$

$y(x) = Y(Z)$



$$\frac{dy}{dx} = \frac{dy}{dZ}\cdot\frac{dZ}{dx}$$

$$\frac{dy}{dx} = \frac{dY}{dZ} \times \frac{-1}{\cdots}$$

Z $\quad x=1$

$$\frac{dy}{dx} = \frac{dY}{dZ} \times \frac{-1}{\delta}$$

$$\frac{d^2y}{dx^2} = \frac{1}{\delta^2} \cdot \frac{d^2Y}{dZ^2}$$

$$\Rightarrow \quad \epsilon \cdot \frac{1}{\delta^2} \frac{d^2Y}{dZ^2} - \frac{1}{\delta}\frac{dY}{dZ} = e^{-(1-\delta Z)}$$

$$\frac{\epsilon}{\delta}\frac{d^2Y}{dZ^2} - \frac{dY}{dZ} = \delta e^{-(1-\delta Z)}$$

Natural choice for δ is $\delta = \epsilon$.

$$\Rightarrow \boxed{\frac{d^2Y}{dZ^2} - \frac{dY}{dZ} = \epsilon e^{-(1-\epsilon Z)}}$$

Let $Y(Z) = Y_0(Z) + \epsilon Y_1(Z) + \dots$

$\underline{\underline{O(1)}}$:- $$\frac{d^2Y_0}{dZ^2} - \frac{dY_0}{dZ} = 0$$

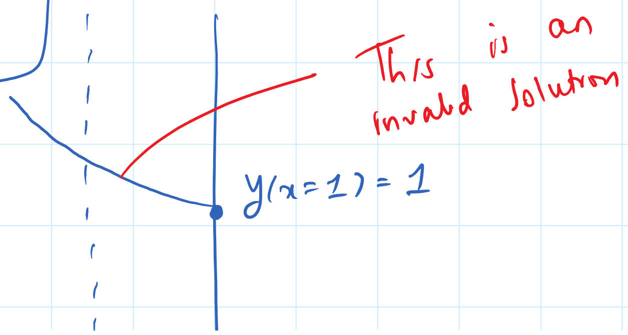
$$\Rightarrow \boxed{Y_0(Z) = c_3 e^{Z} - c_4}$$

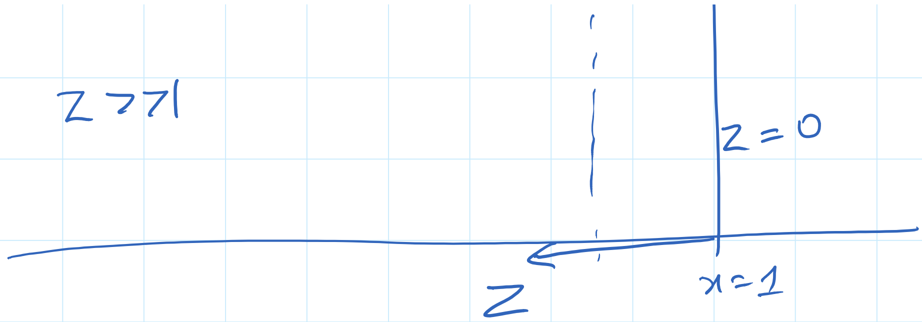
$$Y_0(Z=0) = 1 \quad \Rightarrow \quad 1 = c_3 - c_4$$

$$\Rightarrow \quad c_4 = c_3 - 1$$

$$\therefore \quad Y_0(Z) = c_3 e^{Z} - (c_3 - 1)$$

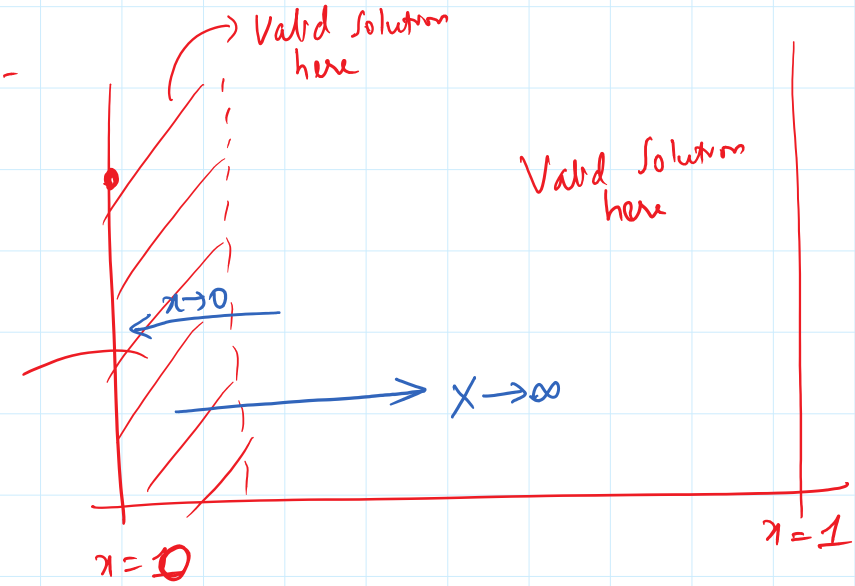
$$\boxed{Y_0(Z) = c_3(e^{Z} - 1) + 1}$$





Since $|Y(Z)|$ grows unboundedly as $Z \to \infty$, we can never match it to the solution near $x = 0$.

<u>Full picture:-</u>



Near $x=0$: $$Y_0(X) = e^{-X} + C_1(1-e^{-X})$$

Away from $x=0$: $$y_0(x) = C_0 - e^{-x}$$

This has to satisfy the boundary condition near $x=1$

$$\Rightarrow y_0(x) = 1 + \frac{1}{e} - e^{-x}$$

<u>Matching the inner & outer solutions:-</u>

We use van Dyke's matching principle.

We want to match the inner region solution to the outer region solution. This obtained as follows:-

$$\lim_{X \to \infty} Y^{inner}(X) = \lim_{x \to 0} y^{outer}(x)$$

or more generally:

$$\lim_{X \to \text{outer region}} Y^{inner}(X) = \lim_{x \to \text{towards inner region}} y^{outer}(x)$$

$$\Rightarrow \lim_{X \to \infty} e^{-X} + c_1(1 - e^{-X}) = \lim_{x \to 0} 1 + \frac{1}{e} - e^{-x}$$

$$\Rightarrow c_1 = \frac{1}{e}$$

<u>Inner Solution</u>:- $Y_0(X) = e^{-X} + \frac{1}{e}(1 - e^{-X})$

<u>Outer Solution</u>:- $y_0(x) = 1 + \frac{1}{e} - e^{-x}$

<u>Composite</u> or <u>Uniformly valid solution</u>:-

$$y_{unif}(x) = Y^{inner}(X) + y^{outer}(x) - y^{overlap}$$

$y^{overlap}$ is the valid where the two functions meet, i.e.,

$$y^{overlap} = \lim Y^{inner} = \lim_{x \to 0} y^{outer}$$

$$y^{overlap} = \lim_{X \to \infty} y^{inner} = \lim_{x \to 0} y$$

$$= \frac{1}{e}$$

$$\therefore \quad y_{unif}(x) = \left\{ e^{-X} + \frac{1}{e}\left(1 - e^{-X}\right) \right\} + \left(1 + \cancel{\frac{1}{e}} - e^{-x}\right) - \cancel{\frac{1}{e}}$$

$$\boxed{y_{unif}(x) = e^{-x/\epsilon} + \frac{1}{e}\left(1 - e^{-x/\epsilon}\right) + 1 - e^{-x}}$$