1D Maps

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We will briefly look at dynamical systems which are represented as difference equations or recursive relations or simply MAPS. In these systems, time is discrete and there is a mapping between previous state of the system to its current state, typically in the form

$$x_{n+1} = f(x_n)$$

where f is a smooth function.

Fixed Points:- If x^* satisfies $f(x^*) = x^*$, then x^* is said to be a fixed point, since if $x_n = x^*$, then

$$x_{n+1} = f(x_n) = f(x^*) = x^*$$

& $$x_{n+2} = f(x_{n+1}) = f(x_*) = x^*$$

$\Rightarrow$ Orbit remains at x^* for all future iterations.

Stability of fixed Points:- To consider stability of x^*, we consider a nearby orbit $x_n = x^* + \eta_n$ and see if the new orbit is attracted or repelled from x^*.

$$\left[\begin{array}{l} x_n = x^* + \eta_n \\ x_{n+1} = x^* + \eta_{n+1} \\ x_{n+1} = f(x_n) \end{array}\right]$$

Substituting this yields

$$x_{n+1} = x^* + \eta_{n+1} = f(x^* + \eta_n)$$
$$= f(x^*) + \eta_n f'(x^*) + O(\eta_n)^2$$

Since $f(x^*) = x^*$, we get

$$x^* + \eta_{n+1} = x^* + \eta_n f'(x^*) + O(\eta_n^2)$$

$$\Rightarrow \boxed{\eta_{+1} = \eta_n f'(x^*)} \quad \text{neglecting } O(\eta_n^2) \text{ terms.}$$

This is a new map with eigenvalue $\lambda = f'(x^*)$

$$\Rightarrow \eta_{n+1} = \lambda \eta_n$$

$$\therefore \eta_1 = \lambda \eta_0$$

$$\eta_2 = \lambda \eta_1 = \lambda(\lambda \eta_0) = \lambda^2 \eta_0$$

⋮

$$\boxed{\eta_n = \lambda^n \eta_0}$$

If $|\lambda| = |f'(x^*)| < 1$, then $\eta_n \to 0$ as $n \to \infty$, which makes x^* to be linearly stable.

Conversely, if $|f'(x^*)| > 1$, then fixed point x^* is unstable.

What we have obtained above is only the linearised stability, but it is known to hold even for the original nonlinear map. But if we have a marginal state when $|f'(x^*)| = 1$, then linear stability cannot tell us much, we the $O(\eta_n^2)$ terms neglected will become crucial for local stability.

Example:- Find fixed points of the map $x_{n+1} = x_n^2$ & determine their stability.

(A) Fixed points satisfy $f(x^*) = x^*$ $\qquad x_{n+1} = f(x_n) = x_n^2$

$$\Rightarrow x^{*2} = x^*$$

$$\Rightarrow x^*(x^*-1) = 0 \Rightarrow \underline{\underline{x^* = 0}}, \; \underline{\underline{x^* = 1}}$$

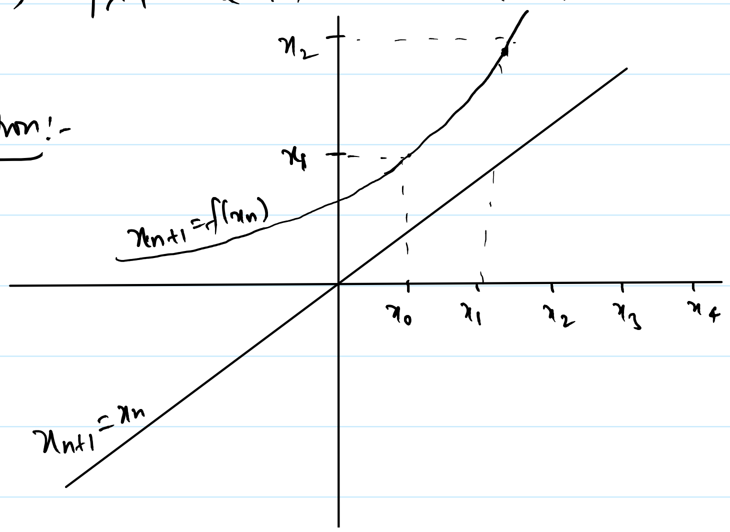
Multiplier or eigenvalue is

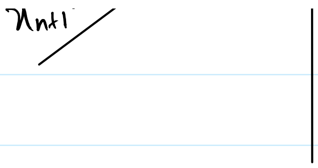
$$\lambda = f'(x^*) = 2x^*$$

for $x^* = 0$; $|\lambda| = 0 < 1$: $x^* = 0$ is stable

for $x^* = 1$; $|\lambda| = 2 > 1$: $x^* = 1$ is unstable.

Cobweb Construction:-



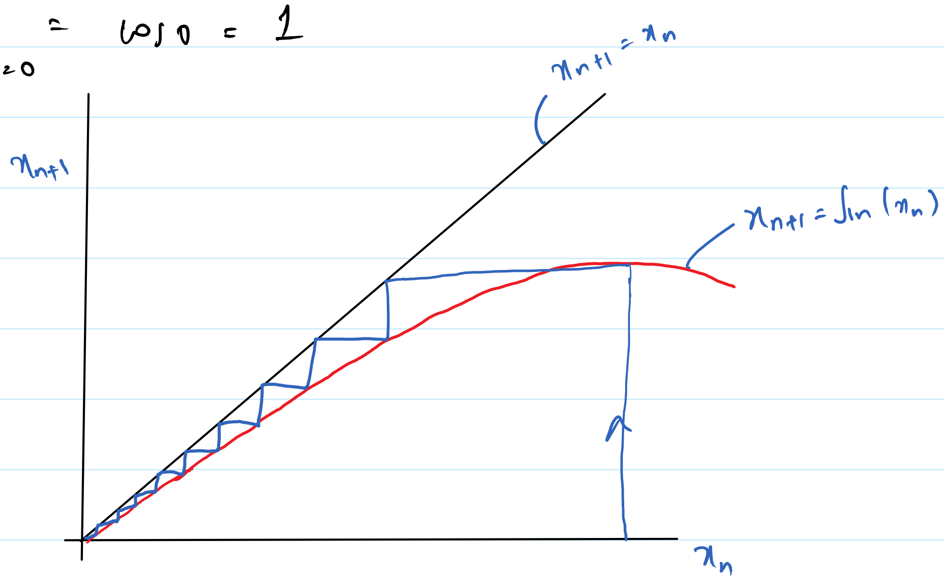


Ex! $x_{n+1} = \sin(x_n)$

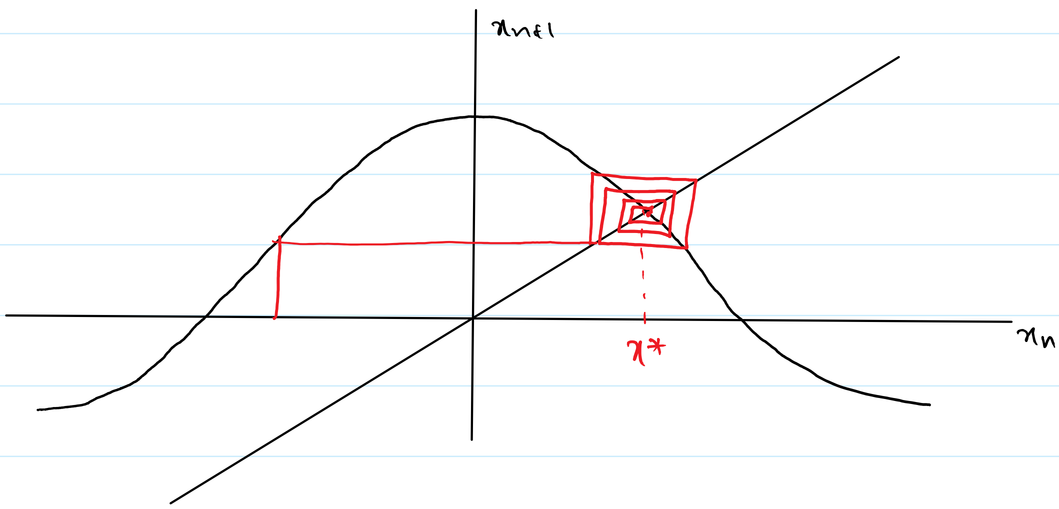
Use cobweb to show stability of $x^* = 0$

$$\lambda = f'(x^*) = \cos x^*$$

$$|\lambda|_{\text{at } x^* = 0} = \cos 0 = 1$$

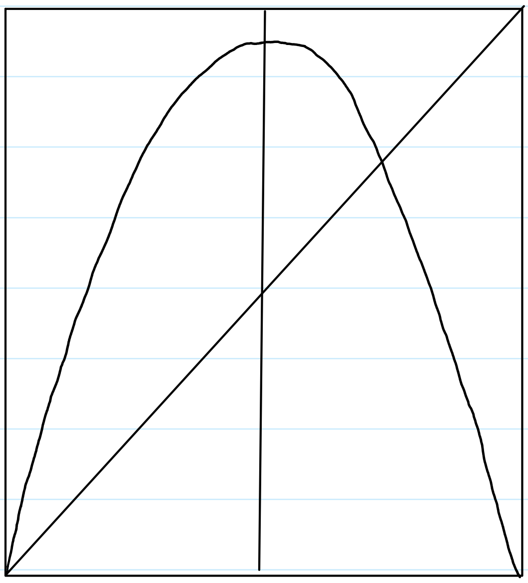


(Q) Given $x_{n+1} = \cos(x_n)$, how does x_n behave as $n \to \infty$.



LOGISTIC MAP

$$x_{n+1} = r x_n (1 - x_n)$$



Logistic Map Analysis :-

$$x_{n+1} = r x_n (1 - x_n) \quad \text{for} \quad 0 \le x_n \le 1 \quad \text{and} \quad 0 \le r \le 4.$$

Find fixed points and their stability :-

Soln: $x^* = f(x^*) = r x^* (1 - x^*)$

$\Rightarrow$ $\underbrace{x^* = 0}_{\text{fixed point for all } r}$ and $\underbrace{x^* = 1 - \frac{1}{r}}_{\text{Allowed only for } r \ge 1.}$ are fixed points.

Stability depends on the multiplier, $f'(x^*) = r - 2 r x^*$

Since $f'(0) = r$, the origin is stable for $r < 1$ and unstable for $r > 1$.

For $x^* = 1 - \frac{1}{r}$; $f'(x^*) = r - 2r \cdot \left(1 - \frac{1}{r}\right)$

$$= r - 2\cancel{r}\left(\frac{r-1}{\cancel{r}}\right)$$

$$= r - 2r + 2 = 2 - r$$

Hence $x^* = 1 - \frac{1}{\lambda}$ is stable for $-1 < (2-\lambda) < 1$

ie; $1 < \lambda < 3$. It is unstable for $\lambda > 3$.