Kinematics - 2

Harish N Dixit
ME5310: Incompressible Fluid Flow
Department of Mechanical & Aerospace Engineering
Indian Institute of Technology Hyderabad
www.iith.ac.in/~hdixit

1 Analysis of relative motion near a point

The force exerted by one portion of a fluid on another depends on the way the fluid deformed by the motion.

Let $\mathbf{u}(\mathbf{x}, t)$ be the velocity at position $\mathbf{x}$ and time t. Then the velocity at a neighbouring point $(\mathbf{x} + \mathbf{r})$ is $\mathbf{u} + \delta\mathbf{u}$. Using Taylor series expansion about $\mathbf{x}$, we get

$$u_i(\mathbf{x} + \mathbf{r}) = u_i(\mathbf{x}) + \frac{\partial u_i}{\partial x_j} r_j + O(r^2). \tag{1}$$

Therefore

$$\begin{aligned} \delta u_i &= u_i(\mathbf{x} + \mathbf{r}) - u_i(\mathbf{x}), \qquad (2) \\ &= r_j \frac{\partial u_i}{\partial x_j}, \qquad \text{we assume linearity in } \delta u_i \text{ vs } \mathbf{r}. \qquad (3) \end{aligned}$$

We can decompose the velocity gradient tensor into a symmetric, $\mathbf{S}$, and antisymmetric, $\boldsymbol{\omega}$, part.

$$\begin{aligned} \frac{\partial u_i}{\partial r_j} &= \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right) + \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} - \frac{\partial u_j}{\partial x_i}\right), \qquad (4) \\ &= S_{ij} + \Omega_{ij}. \qquad (5) \end{aligned}$$

Here S_{ij} is the rate-of-strain tensor and Ω_{ij} is the vorticity tensor.

Therefore, the relative velocity becomes

$$\begin{aligned} \delta u_i &= r_j S_{ij} + r_j \Omega_{ij}, \qquad (6) \\ &= \delta u_i^{(s)} + \delta u_i^{(a)}. \qquad (7) \end{aligned}$$

The two contributions are distinct and we will examine each one of them carefully.

2 Symmetric contribution, $\delta u_i^{(s)}$

S_{ij} can be further decomposed into a diagnol tensor (with non-zero trace) and a symmetric traceless tensor. Hence

$$S_{ij} = \frac{1}{3} S_{kk}\delta_{ij} + \left(S_{ij} - \frac{1}{3} S_{kk}\delta_{ij}\right). \tag{8}$$

Hence we have[1]

$$r_j S_{ij} = \frac{\partial \Phi}{\partial x_i}, \tag{9}$$

where $\Phi_1 = \frac{1}{2} r_k r_l S_{kl}$. Note that Φ = constant form a family of quadrics (look up conic sections). Now $\frac{\partial \Phi}{\partial x_i}$ gives the normal direction to the quadrics. Thus, the existence of such a scalar potential implies that the corresponding velocity is directed normal to the constant Φ surfaces.

The nature of $\delta \mathbf{u}^{(s)}$ contribution to $\delta \mathbf{u}$ becomes clearer if we choose the orthogonal axes of reference in such a way that the off-diagnol elements of S_{ij} become zero. The axes of reference then coincide with the principal axes of S_{ij} and the family of quadrics becomes

$$\Phi = \frac{1}{2}\left(a {r'_1}^2 + b {r'_2}^2 + c {r'_3}^2\right), \tag{15}$$

where $r'_i, i = 1, 2, 3$ are the components of $\mathbf{r}$ in the principal direction and a, b, c are the diagonal components of the tensor S'_{ij} such that

$$S'_{ij} = \frac{\partial r_k}{\partial x'_i} \frac{\partial r_l}{\partial x'_j} S_{kl}. \tag{16}$$

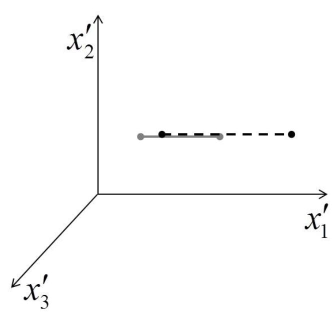
Hence

$$\mathbf{S}' = \begin{pmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{pmatrix}.$$

Note that $a + b + c = S'_{ii} = S_{ii} = \frac{\partial u_i}{\partial x_i}$.

The contribution $\delta \mathbf{u}^{(s)}$ to the relative velocity therefore has three components, (ar'_1, br'_2, cr'_3) with reference to the new axes.

Take the case of a line element initially parallel to x_1 axis. We know that $S'_{11} = \frac{\partial u_1}{\partial x_1} \approx \frac{\Delta u_1}{\Delta x_1}$. Consider two points initially on this line element separated by a distance Δx_1. We therefore have



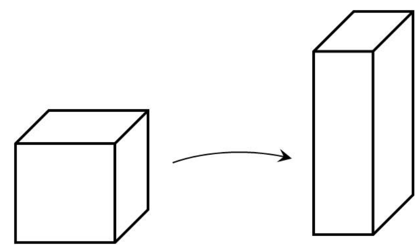
[1]Check: We assumed $\delta \mathbf{u}$ to be linear in $\mathbf{r}$, i.e., we neglected $O(r^2)$ terms in $\delta \mathbf{u}$. Hence

$$\begin{aligned}
\frac{\partial \Phi}{\partial x_i} &= \frac{1}{2}\left(\frac{\partial r_k}{\partial x_i} r_l S_{kl} + r_k \frac{\partial r_l}{\partial x_i} S_{kl} + r_k r_l \frac{\partial S_{kl}}{\partial x_i}\right), && (10) \\
&= \frac{1}{2}\left(r_l S_{il} + r_k S_{ki} + O(r^2)\right), && (11) \\
&= \frac{1}{2}\left(r_l S_{il} + r_k S_{ik}\right), && (12) \\
&= \frac{1}{2}\left(2 r_k S_{ik}\right), \quad \text{since } l \text{ and } k \text{ are dummy variables}, && (13) \\
&= r_j S_{ij}. && (14)
\end{aligned}$$

$$
\begin{aligned}
\Delta u_1 &= u_1(x_1 + \Delta x_1) - u_1(x_1), && (17)\\
&= \frac{\partial u_1}{\partial x_1}\Delta x_1, && (18)\\
&= \Delta x_1 S'_{11}, && (19)\\
&= a\Delta x_1. && (20)
\end{aligned}
$$

If $a > 0$, then the velocity at $(x_1 + \Delta x_1$ is higher than the velocity at x_1. So if $u_1(x_1)$ is positive, then the relative velocity contribution is such that the point $(x_1 + \Delta x_1)$ moves further away from x_1 thus stretching the line element. The rate of this stretching depends on the magnitude of a.
The contribution of $\delta \mathbf{u}^{(s)}$ is said to represent *pure straining motion.*

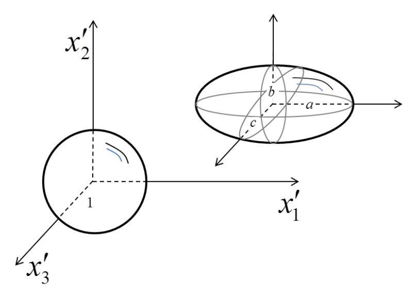
Extending the above argument to all the three directions, we say say that a cube of each side equal to unit length stretches into a cuboid with sides of lengths a, b, c respectively. The volume change in this operation is then equal to $1 - abc$. For an incompressible fluid, since $\Delta \cdot \mathbf{u} = S_{ii} = 0$, $a + b + c = 0$. In such a case, pure straining motion is a volume preserving operation.



The same pure straining motion also extends a sphere of unit radius into an ellipsoid of minor axes of lengths a, b, c. A classic example of pure straining motion is an extensional flow. Experimentally, such a flow is generated with a four roll mill. This is how stretching experiments on bubbles, drops and polymers are carried out in reality.

It is customary to distinguish between straining motions which result in volume changes and those that don't result in any volume change. The latter part would be identical to the case of an incompressible fluid. So, for a general compressible fluid, we can write

$$
\begin{aligned}
\Phi = &\text{ Isotropic expansion accommodating all volume changes}\\
&+ \text{pure straining motion without change in volume.} \qquad (21)
\end{aligned}
$$



This is accomplished by decomposing S_{ij} into an isotropic part with a non-zero trace and a traceless part, i.e.,

$$S_{ij} = \frac{1}{3}S_{kk}\delta_{ij} + \left(S_{ij} - \frac{1}{3}S_{kk}\delta_{ij}\right). \qquad (22)$$

We therefore have

$$\Phi = \frac{1}{6}S_{kk}r^2 + \frac{1}{2}\left(S_{kl} - \frac{1}{3}S_{mm}\delta_{kl}\right)r_k r_l. \qquad (23)$$

It is important to remember that in the above analysis, we have analyzed the straining motions only in the principal directions. We achieved this by tilting the coordinate axes to coincide with the principal directions. As a result, only the diagonal entries of S_{ij} survived. To analyze the role of off-diagonal entries, we will need a a more careful analysis which will be postponed to a later section. But at this point, it suffices to state that S_{ij} contributes to a pure straining motion with or without volume change.

3 Anti-symmetric contribution, $\delta u_i^{(a)}$

We see that Ω_{ij} is anti-symmetrical. We can therefore write

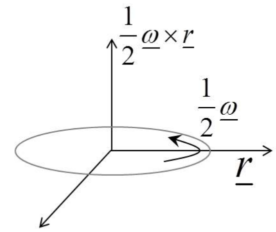
$$\Omega_{ij} = -\frac{1}{2}\epsilon_{ijk}\omega_k. \qquad (24)$$

The negative $-1/2$ is there for convenience. We therefore have

$$\delta u_i^{(a)} = r_j \Omega_{ij} = -\frac{1}{2}\epsilon_{ijk} r_j \omega_k, \quad (25)$$

$$= \frac{1}{2}(\boldsymbol{\omega} \times \mathbf{r})_i. \quad (26)$$

Clearly, $\delta u_i^{(a)}$ is the tangential velocity produced at a point with position vector $\mathbf{r}$ resulting in a solid-body rotation with angular velocity $\boldsymbol{\omega}/2$.
Comparing the components of Ω_{ij} with the last expression, we can write the corresponding components of $\boldsymbol{\omega}$.



$$\omega_1 = \frac{\partial u_3}{\partial x_2} - \frac{\partial u_2}{\partial x_3}, \quad (27)$$

$$\omega_2 = \frac{\partial u_1}{\partial x_3} - \frac{\partial u_3}{\partial x_1}, \quad (28)$$

$$\omega_3 = \frac{\partial u_2}{\partial x_1} - \frac{\partial u_1}{\partial x_2}. \quad (29)$$

The vector $\boldsymbol{\omega}$ is called the *local vorticity*. In symbolic notation, we can also write

$$\boldsymbol{\omega} = \boldsymbol{\nabla} \times \mathbf{u}. \quad (30)$$

If $\boldsymbol{\nabla} \times \mathbf{u} = 0$ everywhere in the flow, we refer to such a flow as an *irrotational flow* since the local rotation vanishes at every point. We will return to this point later.

3.1 Additional physical interpretation

It is easy to see why $\boldsymbol{\nabla} \times \mathbf{u}$ appears as twice the local angular velocity of the fluid. By using Kelvin-Stokes theorem (also simply stated as the Stokes theorem), we have

$$\int (\boldsymbol{\nabla} \times \mathbf{u}) \cdot \mathbf{n} dA = \oint \mathbf{u} \cdot \mathbf{dr}. \quad (31)$$

Let the area be a small circle of radius a. We therefore have

$$\text{Tangential velocity averaged over circumference} = \frac{1}{2\pi a}\oint \mathbf{u} \cdot \mathbf{dr}, \quad (32)$$

$$= a\frac{1}{2\pi a^2}\oint \mathbf{u} \cdot \mathbf{dr}, \quad (33)$$

$$= a\frac{1}{2}(\boldsymbol{\nabla} \times \mathbf{u}) \cdot \mathbf{n}. \quad (34)$$

In the last expression, we have assumed $(\boldsymbol{\nabla} \times \mathbf{u})$ to be constant since it is reasonable to assume that the vorticity field is constant over a very tiny area of size πa^2.

But tangential velocity at the edge of a circle of radius a is given by $u_t = a \times$ (angular velocity). Hence

$$\text{angular velocity} = \frac{1}{2}(\boldsymbol{\nabla} \times \mathbf{u}). \quad (35)$$

4 Summary

In summary, we have seen that, to first order in linear dimensions of a small region surrounding the position $\mathbf{x}$, the velocity in this region consists, in effect, of a superposition of

1. a *uniform translation* with velocity $\mathbf{u}(x)$,

2. a *pure straining* motion characterized by a rate-of-strain tensor, S_{ij}, which itself can be further decomposed into an isotropic expansion and a straining motion without change in volume,

3. a *rigid-body rotation* with angular velocity $\dfrac{\boldsymbol{\omega}}{2}$.

In analytical terms, the conclusion is that the velocity at the position $(\mathbf{x}+\mathbf{r})$ may be written approximately as

$$u_i(\mathbf{x}+\mathbf{r}) = u_i(\mathbf{x}) + \frac{\partial}{\partial x_i}\left(\frac{1}{2}r_j r_k S_{jk}\right) + \frac{1}{2}\epsilon_{ijk}\omega_j r_k + O(r^2), \tag{36}$$

or in symbolic notation as

$$\mathbf{u}(\mathbf{x}+\mathbf{r}) = \mathbf{u}(\mathbf{x}) + \frac{\partial \Phi}{\partial \mathbf{x}} + \frac{1}{2}(\boldsymbol{\omega}\times\mathbf{r}) + O(r^2), \tag{37}$$

where S_{ij} and ω_j are evaluated at the point $\mathbf{x}$.

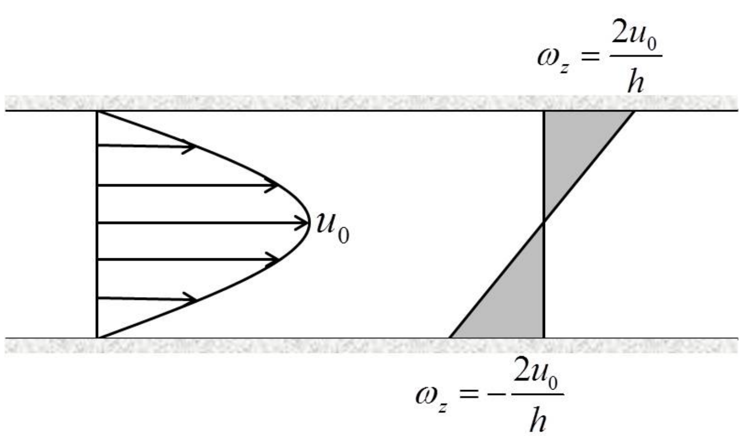
5 Vorticity and vortex

We defined the local vorticity of a fluid as $\boldsymbol{\omega} = \boldsymbol{\nabla}\times\mathbf{u}$. We briefly discuss the difference between vorticity and vortex here by considering three different flow cases.

5.1 Unidirectional shear flow

Consider a simple viscous flow through a channel of width $2h$. The velocity is given by

$$[u, v, w] = \left[u_0\left(1-\frac{y^2}{h^2}\right), 0, 0\right]. \tag{38}$$



The only non-zero velocity component is

$$\omega_z = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}, \tag{39}$$

$$= 0 - u_0 \left(\frac{-2y}{h^2} \right), \tag{40}$$

$$= \frac{2u_0}{h} \frac{y}{h}. \tag{41}$$

Therefore $|\omega_z|$ is maximum at $y = \pm h$ and $\omega_z = 0$ at $y = 0$. Here the streamlines are straight, yet vorticity is non-zero.

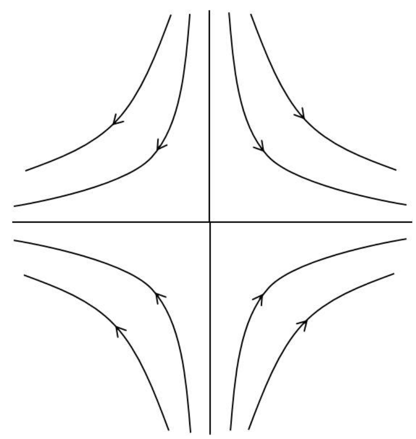
5.2 Stagnation point flow

Now let us consider another flow as shown schematically in figure XXXX. The velocity components are given by

$$(u, v, w) = (cx, -cy, 0), \tag{42}$$

where c is an arbitrary positive constant. It is easy to verify that all the vorticity components are zero, i.e.,

$$\omega_x = \omega_y = \omega_z = 0. \tag{43}$$



The streamlines in this flow are curved, yet the vorticity is zero. The present example clearly demonstrates that vorticity is not directly connected with the curvature of streamlines.

5.3 Point vortex

A point vortex is an irrotational vortex with vorticity at just one single point in the entire flow. The velocity components in the cylindrical polar coordinates are given by

$$(u_r, u_\theta, u_z) = \left(0, \frac{\Gamma}{r}, 0 \right). \tag{44}$$

The z component of vorticity in cylindrical coordinates is given by

$$\omega_z = \frac{1}{r} \frac{\partial (r u_\theta)}{\partial r} - \frac{1}{r} \frac{\partial u_r}{\partial \theta}, \tag{45}$$

$$= \frac{1}{r} \frac{\partial \Gamma}{\partial r} = 0, \qquad \text{except at } r = 0. \tag{46}$$

Vorticity is identically zero everywhere except at $r = 0$ where the derivative operation is not well defined. This is an example of a vortex with zero vorticity, and a flow with closed streamlines with zero vorticity. We will return to the subject of irrotational vortex later when dealing with the concept of circulation.

6 Decomposition of straining motion

In the previous sections, we decomposed the total deformation into a translation, a pure straining motion and a solid-body rotation. The pure straining motion was analyzed by rotating the coordinate axes to coincide with the principal axes such that only the diagonal entries of S_{ij} survive.

The rate-of-strain tensor[2], $S_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right)$, is a second order tensor. Written in a matrix form, we have

$$\mathbf{S} = \begin{pmatrix} S_{11} & S_{12} & S_{13} \\ S_{12} & S_{22} & S_{23} \\ S_{13} & S_{23} & S_{33} \end{pmatrix}.$$

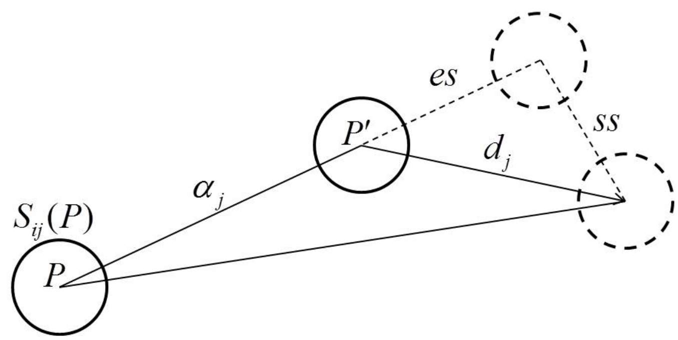
We will show that the diagonal entries in $\mathbf{S}$ are associated with *extensional strains* and off-diagonal extras are associated with *shear strains*. We know that the straining component of relative velocity is given by

$$\delta u_i^{(s)} = r_j S_{ij}. \tag{47}$$

Let $\mathbf{r} = \boldsymbol{\alpha} ds$ where $\boldsymbol{\alpha}$ is a unit vector in the $\mathbf{r}$ direction and ds is the length. Hence

$$\delta u_i^{(s)} = \alpha_j S_{ij} ds = d_i ds, \tag{48}$$

where $\mathbf{d} = \boldsymbol{\alpha} \cdot \mathbf{S}$ is the strain vector as shown schematically in the figure.



For a point P' in the α_i direction from P, the vector d_i is the strain rate of P' with respect to P. It indicates the direction and velocity with which P' moves away from P. Clearly, as shown in the figure, d_i is not necessarily along α_i. As suggested by vector decomposition, we can write

$$\delta u_i^{(s)} = \delta u_i^{(es)} + \delta u_i^{(ss)}. \tag{49}$$

The first term on RHS is the extensional strain which denotes deformation in the α_i direction, whereas the second term is the shearing strain which denotes deformation perpendicular to the α_i direction.

6.1 Extensional strain

The extensional strain is proportional to $\boldsymbol{\alpha} \cdot \mathbf{d}$ pointing in the $\boldsymbol{\alpha}$ direction. This can be written as

$$\delta \mathbf{u}^{(es)} = \boldsymbol{\alpha}(\boldsymbol{\alpha} \cdot \mathbf{d}) ds, \tag{50}$$

$$\delta u_i^{(es)} = \alpha_i \ \alpha_j d_j \ ds. \tag{51}$$

For example, let P be at the origin and P' be on the x axis at a distance ds. The vector $\mathbf{r} = \boldsymbol{\alpha} ds$ is therefore pointing in the direction of x axis and we have

$$\alpha_1 = 1, \quad \alpha_2 = \alpha_3 = 0.$$

Hence

$$\begin{aligned} \delta u_1^{(es)} &= d_1 ds, \\ &= (\boldsymbol{\alpha} \cdot \mathbf{S})_1 ds, \\ &= S_{11} ds. \end{aligned} \tag{52}$$

Therefore S_{11} is the extension rate of two particles separated in the x_1 direction.
Similarly, it can be shown that $\delta u_2^{(es)} = \alpha_2(\alpha_j d_j) ds = 0$.

[2] S_{ij} is also referred to as the rate-of-deformation tensor. In literature, this tensor is also referred to with other symbols, ε_{ij} or $\dot{\gamma}_{ij}$.

6.2 Shear strain

The shear strain is the component of strain vector, d_i perpendicular to the α_i direction. This can be accomplished by the cross product of $\boldsymbol{\alpha}$ and $(\boldsymbol{\alpha} \times \mathbf{d})$. Therefore, we have

$$\delta \mathbf{u}^{(ss)} = (\boldsymbol{\alpha} \times \mathbf{d}) \times \boldsymbol{\alpha} \; ds. \tag{53}$$

Alternately, we can also calculate the shear strain as

$$\delta \mathbf{u}^{(ss)} = \delta \mathbf{u}^{(s)} - \delta \mathbf{u}^{(es)}. \tag{54}$$

If $P = (0, 0, 0)$ and $P' = (ds, 0, 0)$, we have $\boldsymbol{\alpha} = (1, 0, 0)$. Hence

$$\boldsymbol{\alpha} \times \mathbf{d} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 0 & 0 \\ d_1 & d_2 & d_3 \end{vmatrix} = -\mathbf{j} \; d_3 + \mathbf{k} \; d_2.$$

and

$$(\boldsymbol{\alpha} \times \mathbf{d}) \times \boldsymbol{\alpha} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & -d_3 & d_2 \\ 1 & 0 & 0 \end{vmatrix} = \mathbf{j} \; d_2 + \mathbf{k} \; d_3.$$

Moreover, since $d_i = \alpha_j S_{ij}$, we have

$$d_1 = \sum_j \alpha_j S_{1j} = S_{11}, \tag{55}$$

$$d_2 = \sum_j \alpha_j S_{2j} = S_{12}. \tag{56}$$

Putting everything together, we have

$$\delta u_1^{(ss)} = 0, \qquad \text{(since there is no } i\text{-component in } (\boldsymbol{\alpha} \times \mathbf{d}) \times \boldsymbol{\alpha}), \tag{57}$$

$$\delta u_2^{(ss)} = d_2 = S_{12}, \tag{58}$$

$$\delta u_3^{(ss)} = d_3 = S_{13}. \tag{59}$$

Clearly, there is no relative velocity in the x_1-direction. S_{12} gives the shearing velocity in the x_2-direction of a particle P' which is originally separated from P only in the x_1-direction. Figure XXXXX

In general, the off-diagonal entries of S_{ij} give the shearing velocity in the j-direction for two particles initially separated in the i-direction.

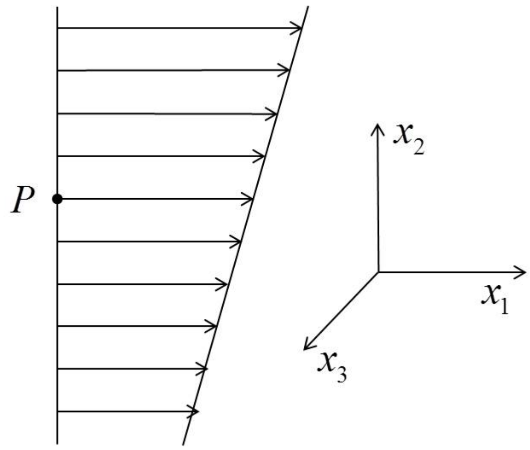
7 Simple Shear Flow

To see the application of all the previous analysis, let us consider a standard flow and determine the components of the relative velocity in the neighborhood of single point, P. Let the velocity components be given by

$$(u_1, u_2, u_3) = (cx_2, 0, 0), \tag{60}$$

where $c > 0$ determines the velocity gradient of the shear flow. This flow is called the *simple shear flow*. The rate-of-strain tensor becomes

$$\mathbf{S} = \begin{pmatrix} 0 & c/2 & 0 \\ c/2 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$



Using $d_i = \alpha_j S_{ij}$, we have

$$d_1 = \alpha_1 S_{11} + \alpha_2 S_{12} + \alpha_3 S_{13} = \alpha_2 \frac{c}{2}, \tag{61}$$

$$d_2 = \alpha_1 S_{21} + \alpha_2 S_{22} + \alpha_3 S_{33} = \alpha_1 \frac{c}{2}, \tag{62}$$

$$d_3 = \alpha_1 S_{31} + \alpha_2 S_{32} + \alpha_3 S_{33} = 0. \tag{63}$$

Vorticity, $\boldsymbol{\omega} = (0, 0, -c)$.

Now we are in a position to calculate the elementary motions of P' with respect to P. Specifically, we consider five points around $P = (0, 0, 0)$ whose vertices and $\boldsymbol{\alpha}$ components are given below:

$$\begin{array}{lll}
\text{At 1:} & P' = (1, 0, 0); & (\alpha_1, \alpha_2, \alpha_3) = (1, 0, 0), \\
\text{At 2:} & P' = (\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0); & (\alpha_1, \alpha_2, \alpha_3) = (\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0), \\
\text{At 3:} & P' = (0, 1, 0); & (\alpha_1, \alpha_2, \alpha_3) = (0, 1, 0), \\
\text{At 4:} & P' = (-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0); & (\alpha_1, \alpha_2, \alpha_3) = (-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0), \\
\text{At 5:} & P' = (-1, 0, 0); & (\alpha_1, \alpha_2, \alpha_3) = (-1, 0, 0).
\end{array}$$

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7.1 The velocity due to solid-body rotation

$$\delta u_i^{(a)} = -\frac{1}{2}\epsilon_{ijk} r_j \omega_k, \tag{64}$$

$$= -\frac{1}{2}\epsilon_{ijk} \alpha_j \omega_k ds, \tag{65}$$

$$= -\frac{1}{2}\epsilon_{ij3} \alpha_j \omega_3 ds \qquad \text{(since only 3-component of } \boldsymbol{\omega} \text{ is non-zero.)} \tag{66}$$

Hence

$$\delta u_1^{(a)} = -\frac{1}{2}\epsilon_{123} \alpha_2 \omega_3 ds, \tag{67}$$

$$= -\frac{1}{2}\alpha_2 (-c) ds, \tag{68}$$

$$= \alpha_2 \frac{c}{2} ds. \tag{69}$$

$$\delta u_2^{(a)} = -\alpha_1 \frac{c}{2} ds, \tag{70}$$

$$\delta u_3^{(a)} = 0. \tag{71}$$

At each of the points P', we have,

$$\begin{array}{lll} \text{At 1:} & \delta u_1^{(a)} = 0, & \delta u_2^{(a)} = -\frac{c}{2} ds, \\ \text{At 2:} & \delta u_1^{(a)} = \frac{c}{2\sqrt{2}} ds, & \delta u_2^{(a)} = -\frac{c}{2\sqrt{2}} ds, \\ \text{At 3:} & \delta u_1^{(a)} = \frac{c}{2} ds, & \delta u_2^{(a)} = 0, \\ \text{At 4:} & \delta u_1^{(a)} = \frac{c}{2\sqrt{2}} ds, & \delta u_2^{(a)} = \frac{c}{2\sqrt{2}} ds, \\ \text{At 5:} & \delta u_1^{(a)} = 0, & \delta u_2^{(a)} = \frac{c}{2} ds. \end{array}$$

.
The relative velocity at points around 0 is shown schematically in figure below.

7.2 The velocity due to straining (deforming) motion

The relative velocity due to straining is given by

$$\delta u_i^{(s)} = r_j S_{ij} = \alpha_j ds S_{ij} = \alpha_j S_{ij} ds = d_i ds. \tag{72}$$

Hence

$$\delta u_1^{(s)} = d_1 ds = \alpha_2 \frac{c}{2} ds, \tag{73}$$
$$\delta u_2^{(s)} = d_2 ds = \alpha_1 \frac{c}{2} ds, \tag{74}$$
$$\delta u_3^{(s)} = d_3 ds = 0. \tag{75}$$

We can further decompose the straining motion into an extensional part and a shear part.

7.2.1 Extensional strain

Recall that

$$\delta u_i^{(es)} = \alpha_i \ \alpha_j d_j \ ds. \tag{76}$$

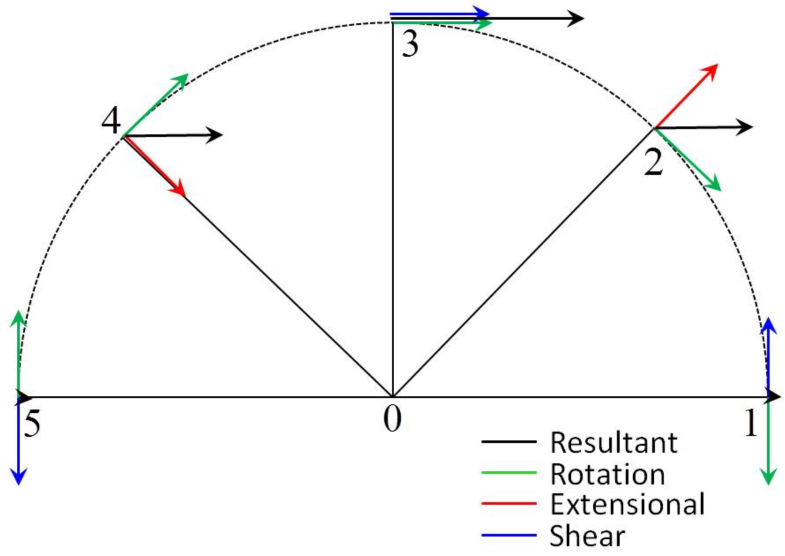
Hence

$$\delta u_1^{(es)} = \alpha_1 \sum_j \alpha_j d_j ds = \alpha_1^2 \alpha_2 c \ ds, \tag{77}$$
$$\delta u_2^{(es)} = \alpha_2 \sum_j \alpha_j d_j ds = \alpha_1 \alpha_2^2 c \ ds, \tag{78}$$
$$\delta u_3^{(es)} = \alpha_3 \sum_j \alpha_j d_j ds = 0. \tag{79}$$

At each of the points P', we have,

$$\begin{array}{lll} \text{At 1:} & \delta u_1^{(es)} = 0, & \delta u_2^{(es)} = 0, \\ \text{At 2:} & \delta u_1^{(es)} = \frac{c}{2\sqrt{2}} ds, & \delta u_2^{(es)} = \frac{c}{2\sqrt{2}} ds, \\ \text{At 3:} & \delta u_1^{(es)} = 0, & \delta u_2^{(es)} = 0, \\ \text{At 4:} & \delta u_1^{(es)} = \frac{c}{2\sqrt{2}} ds, & \delta u_2^{(es)} = -\frac{c}{2\sqrt{2}} ds, \\ \text{At 5:} & \delta u_1^{(es)} = 0, & \delta u_2^{(es)} = 0. \end{array}$$

.
The relative velocity at points around 0 is shown schematically in figure below.



7.2.2 Shear strain

The shear strain can be obtained by removing the extensional part of the total strain:

$$\delta u_i^{(ss)} = \delta u_i^{(s)} - \delta u_i^{(es)}. \tag{80}$$

Therefore,

$$\delta u_1^{(ss)} = \left(\frac{1}{2} - \alpha_1^2\right) \alpha_2 c \; ds, \tag{81}$$

$$\delta u_2^{(ss)} = \left(\frac{1}{2} - \alpha_2^2\right) \alpha_1 c \; ds, \tag{82}$$

$$\delta u_3^{(ss)} = 0. \tag{83}$$

Maximum shearing happens at the coordinate axis, i.e. when $\alpha_1 = 0$ or $\alpha_2 = 0$. When both $\alpha_1 = \pm 1/\sqrt{2}$ and $\alpha_2 = \pm 1/\sqrt{2}$, shearing motion is zero. This forms a set of axis (the principal axes) rotated at 45° from the coordinate axes.

At each of the points P', we have,

$$\begin{array}{lll}
\text{At 1:} & \delta u_1^{(ss)} = 0, & \delta u_2^{(ss)} = \frac{c}{2} ds, \\
\text{At 2:} & \delta u_1^{(ss)} = 0, & \delta u_2^{(ss)} = 0, \\
\text{At 3:} & \delta u_1^{(ss)} = \frac{c}{2} ds, & \delta u_2^{(ss)} = 0, \\
\text{At 4:} & \delta u_1^{(ss)} = 0, & \delta u_2^{(ss)} = 0, \\
\text{At 5:} & \delta u_1^{(ss)} = 0, & \delta u_2^{(ss)} = -\frac{c}{2} ds.
\end{array}$$

.
The relative velocity at points around 0 is shown schematically in figure.

7.2.3 Principal axes

The principal axes of S_{ij} are defined such that only the diagonal entries of S_{ij} survive. The contribution of relative motion with such a rate-of-strain tensor suppresses shear strain contribution as we have already

seen that the shear strain contribution arises from the off-diagonal entries of S_{ij}. Clearly, the shear strain contribution goes to zero at points 2 and 4. At both these points, the total strain is identical to the extensional strain. Therefore, we expect that principal axes to coincide with directions $\vec{02}$ and $\vec{04}$.
To determine the principal axes of S_{ij}, we first calculate the eigenvalues and then its eigenvectors. The characteristic equation of S_{ij} is given by

$$\lambda^3 - I^{(1)}\lambda^2 - I^{(2)}\lambda - I^{(3)} = 0, \tag{84}$$

where $I^{(1)}$, $I^{(2)}$ and $I^{(3)}$ are the invariants of matrix S_{ij}.

$$I^{(1)} = tr(S) = S_{ii}, \tag{85}$$

$$I^{(2)} = \frac{1}{2}\left[tr(\mathbf{S}^2) - (tr(\mathbf{S}))^2\right] = \frac{1}{2}(S_{ij}S_{ji} - S_{ii}S_{jj}), \tag{86}$$

$$I^{(3)} = Det[\mathbf{S}]. \tag{87}$$

In the case of simple shear flow, $S_{12} = S_{21} = c/2$ and all other entries are zeros. The invariants then take the values

$$I^{(1)} = I^{(3)} = 0, \qquad I^{(2)} = \frac{c^2}{4}. \tag{88}$$

The characteristic equation becomes

$$\lambda^3 - \left(\frac{c^2}{4}\right)\lambda = 0, \tag{89}$$

which gives

$$\lambda = 0 \quad \text{or} \quad \lambda = \pm\frac{c}{2}. \tag{90}$$

The first eigenvalue is redundant as the base flow is two-dimensional. The eigenvectors, $\boldsymbol{\beta}$, are obtained from the equation

$$\beta_i\left(S_{ij} - \lambda\delta_{ij}\right) = 0. \tag{91}$$

Inserting $\lambda = c/2$, we get

$$-\frac{c}{2}\beta_1^{(1)} + \frac{c}{2}\beta_2^{(1)} = 0, \tag{92}$$

which is satisfied with $\boldsymbol{\beta}^{(1)} = \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$. Similarly, the second eigenvector becomes $\boldsymbol{\beta}^{(2)} = \left(-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$. Both these vectors in the $\pm 45°$ directions. These principal directions are consistent with the directions of maximum and minimum extensional strain together with zero shear strain.