Bounds on vertex colorings with restrictions on the union of color classes

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Abstract

A proper coloring of a graph is a labeled partition of its vertices into parts which are independent sets. In this paper, given a positive integer j and a family $\mathcal{F}$ of connected graphs, we consider proper colorings in which we require that the union of any j color classes induces a subgraph which has no copy of any member of $\mathcal{F}$. This generalizes some well-known types of proper colorings like acyclic colorings (where $j = 2$ and $\mathcal{F}$ is the collection of all even cycles) and star colorings (where $j = 2$ and $\mathcal{F}$ consists of just a path on 4 vertices), etc. For this type of coloring, we obtain an upper bound of $O\left(d^{\frac{k-1}{k-j}}\right)$ on the minimum number of colors sufficient to obtain such a coloring. Here, d refers to the maximum degree of the graph and k is the size of the smallest member of $\mathcal{F}$. For the case of $j = 2$, we also obtain lower bounds on the minimum number of colors needed in the worst case.

As a corollary, we obtain bounds on the minimum number of colors sufficient to obtain proper colorings in which the union of any j color classes is a graph of bounded treewidth. In particular, using $O(d^{8/7})$ colors, one can obtain a proper coloring of the vertices of a graph so that the union of any two color classes has treewidth at most 2. We also show that this bound is tight within a multiplicative factor of $O((\log d)^{1/7})$.

We also consider generalizations where we require simultaneously for several pairs $(j_i, \mathcal{F}_i)$ $(i = 1, \ldots, l)$ that the union of *any* j_i color classes has no copy of any member of $\mathcal{F}_i$ and obtain upper bounds on the corresponding chromatic numbers.

Keywords : proper colorings, restricted colorings, acyclic colorings, treewidth, forbidden subgraphs, forbidden minors, probabilistic arguments, random graphs.

1 Introduction

All graphs considered here are simple and undirected. A proper vertex coloring of a simple graph $G = (V, E)$ is an assignment of colors to each $v \in V$ so that adjacent vertices do not get the same color. The minimum number of colors used by any proper coloring of a graph G is called its chromatic number and is denoted by $\chi(G)$. It is well-known that $\chi(G) \leq \Delta + 1$ where Δ denotes the maximum degree in G. A *trivial coloring* of G is one in which every vertex gets a distinct color.

Several variants of coloring have been studied. Some of these variants relax the condition that each color class should induce an independent set. The arboricity of a graph, for example, only requires that each color class is a forest. On the other hand, there are variants such as the acyclic coloring which impose restictions on the union of color classes in addition to the requirement of properness.

An acyclic vertex coloring, introduced by Grünbaum (see [12]), is a proper vertex coloring of G such that there are no bichromatic cycles, equivalently, the union of any two color classes must induce a forest. The minimum number of colors used by any such coloring is called the acyclic chromatic number of G and is denoted by $a(G)$. Alon, McDiarmid and Reed [1] showed that $\max\{a(G) : \Delta(G) = d\} = O(d^{4/3})$ and also that this bound is nearly tight.

A further restriction is the star coloring of a graph G which is a proper coloring of $V(G)$ such that no path of length 3 is bicolored. Equivalently, the union of any two color classes is a collection of vertex disjoint stars. This was introduced by Fertin, Raspaud and Reed in [FRR '04] where they obtained bounds for the star chromatic number (defined as the minimum number of colors used by any such coloring) for various classes of graphs as well as a general upper bound of $O(d^{3/2})$ where d denotes the maximum degree of G.

A distance-2 (vertex) coloring of G is a coloring of V so that any two vertices whose distance is at most 2 do not get the same color. In other words, it is a proper coloring of the square G^2 of G. Equivalently, a distance-2 coloring is a proper coloring in which the subgraph induced by the union of any two color classes does not have a path (induced or not) on 3 vertices. The minimum number of colors sufficient for such a coloring is $\chi(G^2)$ and this parameter is closely related to the span of a radio-coloring of a graph [11]. Radiocoloring of a graph has applications in mobile communication.

Each of the above notions of coloring imposes certain restrictions on the union of one or more color classes. In this paper, we consider a notion of coloring which generalizes these notions by requiring that the union of any j (for some fixed $j \geq 1$) color classes forms a subgraph which satisfies a certain (hereditary) property. Such a notion has in fact been implicitly considered by Nešetřil and Ossona de Mendez in [15] where the authors prove the strong result that some of these chromatic numbers are bounded for families of graphs which do not contain a fixed graph as a minor. In this paper, we find upper bounds for these general chromatic numbers in terms of the maximum degree of a graph. The formal definition of the general coloring notion described above is presented below.

Definition 1.1 *Let j be a positive integer and $\mathcal{F}$ be a family of connected graphs of (usual) chromatic number at most j such that for each $H \in \mathcal{F}$, $|V(H)| > j$. We define a $(j, \mathcal{F})$-*subgraph coloring *of G to be a proper coloring of $V(G)$ so that the subgraph of G induced by the union of any j color classes does not contain an isomorphic copy of H as a subgraph, for each $H \in \mathcal{F}$. We denote by $\chi_{j,\mathcal{F}}(G)$ the minimum number of colors used by any $(j, \mathcal{F})$-subgraph coloring of G.*

Remark: We require $j < |V(H)|$ for each $H \in \mathcal{F}$ because otherwise if G contains a copy of H such that $j \geq |V(H)|$, no proper coloring of $V(G)$ would be a $(j, \mathcal{F})$-subgraph coloring. Also if $j < |V(H)|$ for each $H \in \mathcal{F}$, we are guaranteed of at least one $(j, \mathcal{F})$-subgraph coloring, namely, the trivial coloring. We include the condition that the chromatic number of H be at most j because otherwise any proper coloring would forbid H and we can remove such a graph H from $\mathcal{F}$.

We now consider a special type of coloring where we require $\mathcal{F}$ to be a special class of graphs and obtain upper bounds on the corresponding chromatic numbers. These bounds will yield bounds on $\chi_{j,\mathcal{F}}(G)$ as a consequence.

Definition 1.2 *Let j and k be positive integers such that $j \leq k$. We define a (j, k)-*coloring *of G to be a proper coloring of $V(G)$ so that in the subgraph induced by the union of any j color classes, each connected component has size at most k. We denote by $\chi^{con}_{j,k}(G)$ the minimum number of colors used by any (j, k)-coloring of $V(G)$.*

Remark: Note that a (j, k)-coloring is the same as a $(j, \mathcal{F})$-subgraph coloring if we choose $\mathcal{F}$ to be the set of all connected graphs on the set of vertices $U = \{1, \ldots, k + 1\}$. Hence a (j, k)-coloring is a specialized version of the $(j, \mathcal{F})$-subgraph coloring notion. This explains the requirement $j \leq k$ in view of the reasons mentioned in the previous remark.

Similarly, an acyclic coloring (or a star coloring) is the same as a $(2, \mathcal{F})$-subgraph coloring for $\mathcal{F} = \{C_2, C_4, C_6 \ldots\}$ (or $\mathcal{F} = \{P_4\}$) where C_i is a cycle on i vertices and P_4 is a path on four vertices.

First, using probabilistic arguments, we obtain the following upper bound on $\chi^{con}_{j,k}(G)$ of any graph in terms of its maximum degree d, which is one of the main results of our paper.

Theorem 1.3 *Let j, k be given positive integers such that $j \leq k$. Then there exists a constant $C = C(j, k)$ such that for any graph G of maximum degree d, $\chi^{con}_{j,k}(G) \leq C d^{\frac{k}{k+1-j}}$.*

The above theorem immediately leads to an upper bound for $(j, \mathcal{F})$-subgraph colorings.

Corollary 1.4 *Let j be a positive integer and $\mathcal{F}$ be a family of connected graphs of chromatic number at most j. Let k (with $k > j$) denote the size of the smallest graph in $\mathcal{F}$. Then there exists a constant $C = C(j,k)$ such that for any graph G of maximum degree d, $\chi_{j,\mathcal{F}}(G) \leq C d^{\frac{k-1}{k-j}}$.*

By choosing $\mathcal{F} = \{P_4\}$ where P_4 is a path of length 3 on 4 vertices and by noting that a $(2, \mathcal{F})$-subgraph coloring is the same as a star coloring, we notice that the bound of $O\left(d^{3/2}\right)$ for star chromatic number obtained in [10] follows as a consequence of Corollary 1.4. Similarly, by choosing $\mathcal{F} = \{K_{1,\beta+1}\}$, we deduce the bound $O\left(d^{\frac{\beta+1}{\beta}}\right)$ (obtained in [13]) on the minimum number of colors used by any β-frugal coloring (a proper coloring in which each vertex has at most β neighbors in any other color class) of G.

Note that the above bounds are not necessarily tight and we can possibly obtain improvements by making use of the structure of the members of $\mathcal{F}$. For the case $j = 2$, we obtain the following lower bound on the minimum number of colors needed (in the worst-case) for a $(2, \mathcal{F})$-subgraph coloring. For any family $\mathcal{F}$ of connected bipartite graphs, let $\chi_{2,\mathcal{F}}(d)$ denote the maximum value of $\chi_{2,\mathcal{F}}(G)$ of any graph G having maximum degree *at most* d. We also let $\chi_{2,\mathcal{F}}(d, bip)$ to denote this maximum when G is restricted to be bipartite.

Theorem 1.5 *Given a connected bipartite graph H with m edges, we have $\chi_{2,\{H\}}(d) = \Omega\left(\frac{d^{1+\frac{1}{m-1}}}{(\log d)^{1/(m-1)}}\right)$. Hence, for any family $\mathcal{F}$ of connected bipartite graphs, we have $\chi_{2,\mathcal{F}}(d) = \Omega\left(\frac{d^{1+\frac{1}{m-1}}}{(\log d)^{1/(m-1)}}\right)$, where m is the minimum number of edges in any member of $\mathcal{F}$. Also, both asymptotic lower bounds hold true for $\chi_{2,\mathcal{F}}(d, bip)$.*

The proof of this theorem is essentially a generalization of the proof of a similar lower bound for acyclic chromatic numbers obtained in [1].

Remark : For H mentioned before, the above theorem implies that $\chi_{2,\{H\}}(G)$ cannot be bounded by any function $f(\chi(G))$. In fact, the proof of this theorem shows something stronger. For some constant $\epsilon > 0$, for every sufficiently large n, there is a bipartite graph G on n vertices having $\chi_{2,\{H\}}(G) \geq \epsilon n$.

Specializing Theorem 1.5 to $(2, k)$-colorings yields the result below.

Corollary 1.6 $\chi^{con}_{2,k}(d) = \Omega\left(\frac{d^{\frac{k}{k-1}}}{(\log d)^{1/(k-1)}}\right)$.

Thus we see that when $j = 2$, Theorem 1.3 is tight up to polylogarithmic factors. Corollary 1.4 on the other hand is not tight uniformly for every family $\mathcal{F}$, even for the case $j = 2$. For example, in the case of the acyclic chromatic number, we can take $\mathcal{F}$ to be the set of cycles of even length and verify that Corollary 1.4 yields a bound of $O(d^{3/2})$ which we know is not tight. This is not surprising because the proof of Corollary 1.4 does not make use of the structure of the members of $\mathcal{F}$.

Further generalizations : It is also possible to extend our results to more restricted colorings where we require simultaneously for several pairs $(j_i, \mathcal{F}_i)$ $(i = 1, \ldots, l)$ that the union of *any* j_i color classes has no copy of any member of $\mathcal{F}_i$. Similarly, we can require simultaneously for several pairs (j_i, k_i) $(i = 1, \ldots, l)$ that the union of *any* j_i color classes has no connected component of size more than k_i. Our results extend to these generalizations also and in Section 5, we discuss such colorings and present upper bounds on the corresponding chromatic numbers.

Low treewidth colorings

An interesting specialization is to restrict the union of color classes to be a graph of bounded treewidth, from which we obtain the notion of (low) treewidth coloring. This naturally generalizes the acyclic vertex coloring which requires the union of two color classes to have treewidth at most 1. Low treewidth colorings have been studied by DeVos et. al. in [7] where it is proved that, for every fixed H and $j \geq 1$, graphs *without a H-minor* can be vertex-partitioned into a bounded number c_H (which depends only on H) of parts so that for every $k \leq j$, the union of any k of them induce a graph of treewidth at most $k-1$. They also prove the surprising result that any graph having no H-minor can be vertex partitioned into two pieces each inducing a subgraph of bounded treewidth. Our focus will be to obtain bounds for treewidth chromatic numbers for arbitrary graphs in terms of the maximum degree.

Before we go for the definition of treewidth chromatic numbers, we first recall one of many equivalent definitions of the treewidth of a graph. The *treewidth* of a graph G (denoted by $tw(G)$) is the minimum k such that G is a partial k-tree. A *partial k-tree* is any subgraph of a k-tree. A *k-tree* is a graph obtained by starting with a complete graph on $k+1$ vertices and then iteratively adding a new vertex and joining it (by an edge) to each member of some k-clique in the partial graph obtained so far.

Definition 1.7 *Let j, k be positive integers such that $j \leq k+1$. We define a (j,k)-*treewidth (vertex) coloring *of a graph $G = (V, E)$ to be a proper coloring of V such that the subgraph induced by the union of any j color classes has treewidth at most k. We denote by $\chi^{tw}_{j,k}(G)$ the minimum number of colors used by any (j,k)-treewidth coloring of G.*

Remark: We require $j \leq k+1$ because otherwise if G contains a clique on $k+2$ vertices, then no proper coloring of $V(G)$ would be a (j,k)-treewidth coloring. Also if $j \leq k+1$, then the trivial coloring is a (j,k)-treewidth coloring.

We note that a (j,k)-treewidth coloring is the same as a $(j, \mathcal{F})$-subgraph coloring where $\mathcal{F} = \{H : \chi(H) \leq j,\ tw(H) = k+1\}$. Also, an acyclic coloring is the same as a $(2,1)$-treewidth coloring.

In Section 3, we prove that Corollary 1.4 also leads to the following upper bounds (Part (i)) for (j,k)-treewidth colorings. We prove Part (ii) by a direct argument.

Theorem 1.8 *Let j, k be positive integers such that $j \le k+1$. Then,*

- (i) *there exists a constant $C = C(j,k)$ such that for any graph G of maximum degree d, $\chi^{tw}_{j,k}(G) \le Cd^{\frac{kj+1}{kj+1-(j-1)^2}}$. In particular, for each $k \ge 3$, we have $\chi^{tw}_{2,k}(G) \le Cd^{(1+\frac{1}{2k})}$.*
- (ii) *When $j = k = 2$, we have the following better bound $\chi^{tw}_{2,2}(G) = O(d^{8/7})$. This is the minimum number of colors sufficient to ensure that any two color classes induce a graph of treewidth at most 2.*

The lower bound given by Theorem 1.5 leads to the following lower bound for treewidth colorings for the case $j = 2$. The proof is provided in Section 3. We use $\chi^{tw}_{2,k}(d)$ to denote the maximum value of $\chi^{tw}_{2,k}(G)$ of any graph G having maximum degree *at most* d.

Theorem 1.9 *For any given $k \ge 2$, $\chi^{tw}_{2,k}(d) = \Omega\left(\frac{d^{1+\frac{2}{k^2+5k}}}{(\log d)^{\frac{2}{k^2+5k}}}\right)$.*

Note that for $k = 2$, the above result implies that the upper bound in Part (ii) of Theorem 1.8 is tight up to polylogarithmic factors.

Minor-free colorings

By restricting the union of color classes to be a graph which is free of H-minor for each member $H \in \mathcal{F}$ for some collection $\mathcal{F}$ of graphs, we obtain the notion of minor-free coloring. Let us recall the definition of minors. A graph H is a *minor* of G if H can be obtained from G by a sequence of zero or more operations each of which is either a contraction of an edge or removal of an edge or removal of a vertex. G *has a H-minor* if some minor of G is isomorphic to H. Otherwise, we say that G is H-minor-free.

Definition 1.10 *Let j be a positive integer and $\mathcal{F}$ be a family of connected graphs such that for each $H \in \mathcal{F}$, $j < |V(H)|$. We define a* $(j,\mathcal{F})$-minor-free coloring *of a graph G to be a proper coloring of $V(G)$ so that the subgraph of G induced by the union of any j color classes is H-minor-free, for every $H \in \mathcal{F}$. We denote by $\chi^{minor}_{j,\mathcal{F}}(G)$ the minimum number of colors used by any $(j,\mathcal{F})$-minor-free coloring of G.*

Remark: We require $j < |V(H)|$ for each $H \in \mathcal{F}$ because otherwise if G contains a copy of H for some $H \in \mathcal{F}$ such that $j \ge |V(H)|$, no proper coloring of $V(G)$ would be a $(j,\mathcal{F})$-minor-free coloring. Also if $j < |V(H)|$ for each $H \in \mathcal{F}$, then the trivial coloring is a $(j,\mathcal{F})$-minor-free coloring.

A $(j, \mathcal{F})$-minor-free coloring is the same as a $(j, \mathcal{F}')$-subgraph coloring where $\mathcal{F}'$ is the set of all j-colorable (in the usual sense) graphs having a minor of some graph $H \in \mathcal{F}$. Also, note that an acyclic coloring is the same as a $(2, \mathcal{F})$-minor-free coloring where $\mathcal{F} = \{K_3\}$. Similarly, a $(2, 2)$-treewidth coloring is the same as a $(2, \mathcal{F})$-minor-free coloring where $\mathcal{F} = \{K_4\}$. This follows from the fact that any graph has treewidth at least 3 if and only if it has K_4 as a minor (see Chapter 12 of [8]).

In general, graphs of treewidth at most k are minor-closed and hence it follows from the Graph Minor Theorem of Robertson and Seymour [17] that such graphs are characterized by a finite set of forbidden minors. Hence, every (j, k)-treewidth coloring is the same as a $(j, \mathcal{F}_k)$-minor-free coloring where $\mathcal{F}_k$ is the set of all minimal minors forbidden for graphs of treewidth at most k.

Applying Corollary 1.4 to the family $\mathcal{F}'$ of j-colorable graphs having a minor of some $H \in \mathcal{F}$, we deduce the following theorem.

Theorem 1.11 *Let j and $\mathcal{F}$ be as defined in Definition 1.10. Let $\mathcal{F}'$ be the family of connected, j-colorable, minimal graphs having a minor of some $H \in \mathcal{F}$. Let $k > j$ be the minimum size $|V(H)|$ of any $H \in \mathcal{F}'$. Then, for any graph G of maximum degree d, $\chi_{j,\mathcal{F}}^{minor}(G) \leq Cd^{\frac{k-1}{k-j}}$ where $C = C(j, k)$.*

By specializing $\mathcal{F}$ to be $\mathcal{F} = \{K_5, K_{3,3}\}$, we get the following corollary. Recall that a graph is planar if and only if it has no minor of either K_5 or $K_{3,3}$. In this case, a $(j, \mathcal{F})$-minor-free coloring is referred to as a $(j, planar)$-coloring and we use $\chi_j^{planar}(G)$ to denote the corresponding chromatic number.

Corollary 1.12 *Let $j \leq 4$ be a positive integer. Then, for any graph G with maximum degree d, we have (i) $\chi_j^{planar}(G) = \chi(G) \leq d$ if $j = 1$, (ii) $\chi_j^{planar}(G) \leq 25\left(d^{8/7}\right)$ if $j = 2$, (iii) there are graphs of maximum degree d for which $\chi_2^{planar}(G) = \Omega\left(\frac{d^{9/8}}{(\log d)^{1/8}}\right)$ and (iv) $\chi_j^{planar}(G) \leq Cd^{\frac{5}{6-j}}$ if $j = 3, 4$. Here, $C = C(j)$.*

Proof of Corollary 1.12 : For $j = 1$, it follows directly from the definition. For $j = 2$, since any $(2, 2)$-treewidth coloring satisfies the required property, the upper bound follows from the proof of Part (*ii*) of Theorem 1.8. The lower bound for $j = 2$ follows from Theorem 1.5 since any such coloring is a $(2, \{K_{3,3}\})$-subgraph coloring. For $j = 3, 4$, we apply Theorem 1.11. ∎

From previous discussions, we see a hierarchy among the generalizations of graph coloring notions introduced in this paper. An acyclic coloring is the same as a $(2, 1)$-treewidth coloring, a (j, k)-treewidth coloring is the same as a $(j, \mathcal{F}_k)$-minor-free coloring ($\mathcal{F}_k$ defined before) and a $(j, \mathcal{F})$-minor-free coloring is the same as a $(j, \mathcal{F}')$-subgraph coloring for suitably defined $\mathcal{F}'$. Because of this hierarchy, if we obtain upper bounds on the chromatic number for $(j, \mathcal{F})$-subgraph coloring, it also leads to upper bounds on the chromatic numbers for other types of colorings. Hence, we focus on obtaining upper bounds on the chromatic numbers of $(j, \mathcal{F})$-subgraph coloring.

Section 2 proves one of our main results, Theorem 1.3 on (j, k)-coloring. Section 3 proves the other main result, namely, the upper bounds on treewidth colorings given by Theorem 1.8. Section 4 proves lower bounds on various colorings. In Section 5, we discuss the generalizations to forbidding several families simultaneously. In Section 6, we present a new bound on $\chi_{j,k}^{con}(G)$ in terms of another invariant which can possibly lead to improved estimates on these chromatic numbers. Finally, we conclude in Section 7 with some remarks and open problems.

2 Proof of results

The proof of Theorem 1.3 is based on probabilistic arguments. We need the following non-symmetric form of Lovász Local Lemma (see [9], [2] and [**?**]).

Lemma 2.1 *Lemma 2.1 Let $\{A_1, A_2, ..., A_n\}$ be a family of events in an arbitrary probability space. Let the graph $H = (V, E)$ on the nodes $1, 2, \ldots, n$ be a dependency digraph for the events A_i; that is, assume that for each i, A_i is mutually independent of the family of events $\{A_j : (i, j) \notin E\}$. If there are reals $0 \le y_i < 1$ such that for all i,*

$$Pr(A_i) \le y_i \prod_{(i,j) \in E} (1 - y_j)$$

then

$$Pr(\cap(\overline{A_i})) \ge \prod_{i=1}^{n} (1 - y_i) > 0$$

so that with positive probability no event A_i occurs.

We prove the following explicit version of Theorem 1.3.

Proposition 2.2 *Let j, k be given positive integers such that $j \le k$. Then for any graph G of maximum degree d, $\chi_{j,k}^{con}(G) < \lceil Cd^{\frac{k}{k+1-j}} \rceil$ where $C = C(j, k) = (4(k+1)(12j)^{k+1})^{\frac{1}{k+1-j}}$.*

Proof Proposition 2.2:

When $j = 1$, a (j, k)-coloring is also a proper coloring and the converse is also true. In this case, $\chi_{1,k}^{con}(G) = \chi(G) \le d + 1 \le Cd$ since $C(1, k) \ge 12$. Hence, without loss of generality, we assume that $j \ge 2$. Now, choose $x = \lceil Cd^{\frac{k}{k+1-j}} \rceil$ where $C = C(j, k) = (4(k+1)(12j)^{k+1})^{\frac{1}{k+1-j}}$.

Let $f : V \to \{1, 2, ..., x\}$ be a random vertex coloring of G, where for each vertex $v \in V$ independently, the color $f(v) \in \{1, 2, ..., x\}$ is chosen uniformly at random. It suffices to prove that with positive probability, f is a (j, k)-coloring of G. To this end, we define a family of bad events whose total failure implies a (j, k)-coloring and use the

Lovász Local Lemma to show that with positive probability none of them occur.

The events we consider are of the following two types.

a) **Type I**: For each pair of adjacent vertices u and v, let $A_{u,v}$ be the event that $f(u) = f(v)$.

b) **Type II**: For every connected induced subgraph L of $V(G)$ such that $|L| = k+1$, let B_L be the event that the vertices in L are colored using at most j colors in the coloring by f.

Now we can see that if none of the events of the two types above occur, then f is a (j,k)-coloring.

Since no event of Type I occurs, the coloring is proper. Since no event of Type II occurs, the union of any j color classes cannot have a connected subgraph on $k+1$ vertices.

It remains to show that with positive probability none of these events happen. To prove this we apply Lemma 2.1. Any event of either of the two types is mutually independent of all events that do not share a vertex in common with the given event.

To enable the application of Local Lemma, we need to estimate the number of events of each type possibly influencing any given event. This estimate is given in the following two simple lemmas. First, we recall the following known fact from [14].

Fact 2.3 *The number of mutually non-isomorphic (or unlabeled) trees on n vertices is at most 4^n.*

Proof This fact is proved in Chapter 8 of [14]. We give an outline of this proof for the sake of completion.

Embed an unlabeled tree in the plane without crossing edges and draw an extra copy of each edge by its side. Fix any vertex with degree one as the root. Start from the root and complete an Eulerian traversal of the edges by always following the rule of traversing the clockwise next edge incident at a vertex. Encode this traversal by representing each edge by a 1 if it takes the traversal to an unvisited vertex and by a 0 otherwise. One can verify that this encoding is an injective one-to-many mapping of unlabeled trees into binary strings of length $2(n-1)$. Since the number of binary strings of length $2(n-1)$ is $4^{n-1} \leq 4^n$, the result is proved. ∎

Lemma 2.4 *Let v be an arbitrary vertex of the graph $G = (V, E)$. Then the following two statements hold.*

(i) v belongs to at most d edges of G.

(ii) v belongs to at most $(k+1)4^{k+1}d^k$ connected induced subgraphs of size $k+1$ in $V(G)$.

Proof of Lemma 2.4

Part (i) follows from the fact that $\Delta(G) = d$.

Part (ii) can be seen as follows: Let $\mathcal{G}(v, k+1)$ be the set of $(k+1)$-element connected induced subgraphs in G containing v and let $\mathcal{T}(v, k+1)$ be the set of $(k+1)$-element trees in G containing v. Each tree in $\mathcal{T}(v, k+1)$ can be identified with a unique connected induced subgraph of G and each connected induced subgraph in $\mathcal{G}(v, k+1)$ has at least one tree in $\mathcal{T}(v,k+1)$ which is identified with it. Thus $|\mathcal{G}(v,k+1)| \leq |\mathcal{T}(v,k+1)|$. We now find an upper bound for $|\mathcal{T}(v,k+1)|$. Since there are at most 4^{k+1} non-isomorphic trees on $k+1$ vertices (by Fact 2.3), there are at most 4^{k+1} choices for choosing the non-isomorphic structure of a tree in $\mathcal{T}(v,k+1)$. Once this is fixed, we now have to embed this tree in G. The number of choices for the position of v in the tree is $k+1$. Now the remaining vertices in the unlabeld tree can be embedded in at most d^k ways. To see this, we observe that there are at most d choices for each neighbor of v in the chosen tree. Once these are fixed, the number of choices for each vertex at distance 2 from v is again at most d. Repeating this process, we can see that the number of choices for embedding all the vertices (other than v) is at most d^k. ∎

Lemma 2.5 *For $i,j \in \{I,II\}$, the (i,j)-th entry of the table given below is an upper bound on the number of events of type j in which can possibly influence an event of type i.*

	I	$II(B_{L'})$
I	$2d$	$2(k+1)4^{k+1}d^k$
$II(B_L)$	$(k+1)d$	$(k+1)^2 4^{k+1}d^k$

The lemma follows from Lemma 2.4 and the fact that any event is mutually independent of all other events which do not share any vertex with the given event.

We now estimate the probability of occurrence of each type of event.

Fact 2.6 *(i) For each type I event A, $Pr(A) = \frac{1}{x}$.*
(ii) For each type II event B, $Pr(B) \leq \frac{j^{k+1}}{x^{k+1-j}}$.

The number of ways in which a $(k+1)$-element set can be colored using at most j colors is at most $\binom{x}{j} j^{k+1} \leq x^j j^{k+1}$. This proves (ii).

We now define the weights y_i to enable us to apply Lemma 2.1.
For an event A of type I, $y_A = \frac{9}{x}$. For an event B of type II, $y_B = \frac{(3j)^{k+1}}{x^{k+1-j}}$. It follows from the definition of x that $y_B < 1$.
By Lemma 2.1, Lemma 2.5 and Fact 2.6, it suffices to verify the following two inequalities.

$$\frac{1}{x} \leq \frac{9}{x}\left(1-\frac{9}{x}\right)^{2d}\left(1-\frac{(3j)^{k+1}}{x^{k+1-j}}\right)^{2(k+1)4^{k+1}d^k} \quad (1)$$

$$\frac{j^{k+1}}{x^{k+1-j}} \leq \frac{(3j)^{k+1}}{x^{k+1-j}}\left(1-\frac{9}{x}\right)^{(k+1)d}\left(1-\frac{(3j)^{k+1}}{x^{k+1-j}}\right)^{(k+1)^2 4^{k+1}d^k} \quad (2)$$

We observe that (2) is equivalent to (1). This can be seen by taking both sides of inequality (2) to the $2/(k+1)$-th power after canceling the term j^{k+1}/x^{k+1-j} on each side. Thus it is sufficient to prove (1).

In (1), we substitute $x = Cd^{\frac{k}{k+1-j}}$ where $C = C(j,k) = (4(k+1)(12j)^{k+1})^{\frac{1}{k+1-j}}$ and using the fact that $(1-\frac{1}{z})^z \geq 1/4$ for all $z \geq 2$, we see that it is sufficient to prove:

$$\frac{1}{9} \leq 4^{-\frac{18d}{x}} 4^{-1/2}$$

Since $x \geq 18d$ for $j \geq 2$, the above inequality is true.

Thus by Lovász Local Lemma, with probability greater than zero none of the bad events occurs and hence there exists a (j,k)-coloring using $O(d^{\frac{k}{k+1-j}})$ colors. This completes the proof of Theorem 1.3. ∎

3 Low treewidth coloring - Proof of Theorem 1.8

We shall show that Part (i) of Theorem 1.8 follows from Corollary 1.4. For this, it only remains to obtain a lower bound on the number of vertices in any j-colorable graph H whose treewidth is at least $k+1$. All such graphs are forbidden for a (j,k)-treewidth coloring. We make use of the following easy to prove observation.

Proposition 3.1 *Let H be a complete j-partite graph $K_{m_1,\ldots,m_j}$ where we assume that $m_1 \leq \ldots \leq m_j$. Then, $tw(H) = m_1+m_2+\ldots+m_{j-1}$.*

Proof of Proposition 3.1 A graph is *chordal* if it has no induced cycle of length 4 or more. A *chordal completion* of a graph $G=(V,E)$ is any super graph $G'=(V,F)$, $E \subseteq F$, which is also chordal. It is well known (see [16]) that the treewidth of a graph G is exactly one less than the minimum value of the maximum clique size $\omega(G')$ of any chordal completion G' of G.

Let $C_1,\ldots,C_j$ be the j color classes of H with $|C_j| = m_j$. Let m denote the sum $m_1+\ldots+m_j$. Any chordal completion H' of H should have enough edges to make each (except possibly one, say, C_i) of the color classes a complete subgraph. Also, to minimize the value of $\omega(H')$, we need to maintain C_i as an independent set in H'. Hence $\omega(H') = m - m_i + 1$. This value is minimized when $i = j$. Our claim follows from this observation. ∎

Proof of Part (i) **of Theorem 1.8:** Fix any j-colorable graph H whose treewidth is at least $k+1$ and has a j-coloring with color classes $C_1,\ldots,C_j$ where we assume, without loss of generality, that $|C_1| \leq \ldots \leq |C_j|$ and also that H is a complete j-partite graph. For each i, let m_i denote $|C_i|$. Then, by the previous proposition, we should have $\sum_{i<j} m_i \geq k+1$ and hence $|V(H)| = \sum_{i\leq j} m_i \geq (k+1)j/(j-1)$. Applying this fact to Corollary 1.4, we obtain (after simplifications) that $\chi^{tw}_{j,k}(G) = O\left(d^{\frac{kj+1}{kj+1-(j-1)^2}}\right)$. This proves Part (i). ∎

For proving Part (ii), we shall need the following well-known result:

Fact 3.2 *([19]) A graph has treewidth at most 2 if and only if it has no subgraph which is isomorphic to a subdivision of K_4.*

We remark that in [6] also, an equivalent statement may be found, where the paper of Wald and Colbourn referred to above is cited.

We now prove part (ii) of Theorem 1.8 in the following specific form.

Proposition 3.3 *Let $G = (V, E)$ be a graph with maximum degree d. Then $\chi^{tw}_{2,2}(G) \leq \lceil 25(d^{8/7}) \rceil$.*

Proof of Proposition 3.3

Put $\alpha = 6/7$; $x = \lceil c_1 c_2 d^{2-\alpha} \rceil$ where $c_1, c_2 > 1$ are constants to be chosen later so that $c_1 c_2 = 25$.

Let $f : V \to \{1, 2, ..., x\}$ be a random vertex coloring of G, where for each vertex $v \in V$ independently, the color $f(v) \in \{1, 2, ..., x\}$ is chosen uniformly at random. It suffices to prove that with positive probability, the union of any two color classes has no subdivision of K_4 and hence has treewidth at most 2. To ensure this, we define a family of bad events which correspond to proper two-colorings of bipartite subdivisions of K_4 in G, then apply the Lovász Local Lemma to show that with positive probability none of them occur, and conclude that since none of them occur f is a (2,2)-treewidth coloring. The events we consider are of the following six types.

a) **Type I**: For each pair of adjacent vertices u and v, let $A_{u,v}$ be the event that $f(u) = f(v)$.

Absence of Type I events ensure properness, so, by Fact 3.2, we need only to ensure each 2-colorable subdivision of K_4 which is present in G is not 2-colored.

To reduce the number of bipartite K_4 subdivisions we need to consider, we use a notion similar to the one employed in [1]. A pair of non-adjacent vertices is called a special pair if they have more than d^α common neighbours.

b) **Type II**: For each pair of special vertices u and v, let $B_{u,v}$ be the event that $f(u) = f(v)$.

If we forbid all events of Types I and II, then it suffices to only ensure that those bipartite K_4 subdivisions are not 2-colored, which do NOT have a triple (u, v, w) such that $\{u, v\}$ forms a special pair and w is one of their common neighbors. This is because any K_4 subdivision having such a triple will be colored with at least 3 colors.

Henceforth, we only focus on bipartite (that is, 2-colorable) K_4 subdivisions which do not have such a triple described before.

Note that every bipartite subdivision of K_4 should have at least 6 vertices. Also note that the graphs H_1, H_2 and $\{H_3, H_4\}$ which we consider below, are the only pairwise non-isomorphic bipartite subdivisions of K_4 on 6,7 and 8 vertices respectively.

c) **Type III**:

For each subgraph $H_1(v_0, v_1, v_2, v_3, v_4, v_5)$ of the form shown below (Figure 1), in which whenever $i = j \pmod 2$, v_i and v_j are non-adjacent and not a special pair, let $C_1\{v_0, v_1, v_2, v_3, v_4, v_5\}$ be the event that H is properly two-colored by f, i.e, $f(v_0) = f(v_2) = f(v_4)$ and $f(v_1) = f(v_3) = f(v_5)$.

d) **Type IV**:

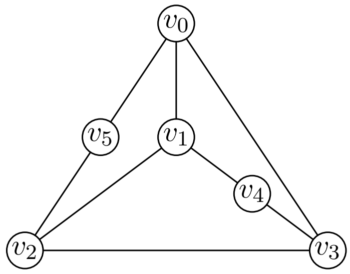

Figure 1: H_1

For each subgraph $H_2(v_0, v_1, v_2, v_3, v_4, v_5, v_6)$ of the form shown below (Figure 2), in which if $i = j \pmod 2$ v_i and v_j are non-adjacent and not a special pair, let $C_2\{v_0, v_1, v_2, v_3, v_4, v_5, v_6\}$ be the event that H is properly two-colored by f, i.e, $f(v_0) = f(v_2) = f(v_4) = f(v_6)$ and $f(v_1) = f(v_3) = f(v_5)$.

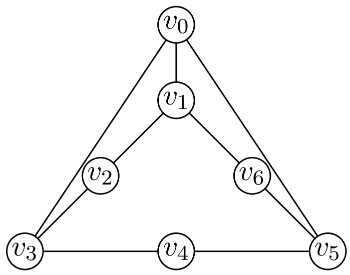

Figure 2: H_2

e) **Type V**:

For each of the two subgraphs $H_3(v_0, v_1, v_2, v_3, v_4, v_5, v_6, v_7)$ and $H_4(v_0, v_1, v_2, v_3, v_4, v_5, v_6, v_7)$ of the forms shown below (Figure 3), in which if $i = j \pmod 2$ v_i and v_j are non-adjacent and not a special pair, let $C_3\{v_0, v_1, v_2, v_3, v_4, v_5, v_6, v_7\}$ be the event that H is properly two-colored by f, i.e, $f(v_0) = f(v_2) = f(v_4) = f(v_6)$ and $f(v_1) = f(v_3) = f(v_5) = f(v_7)$.

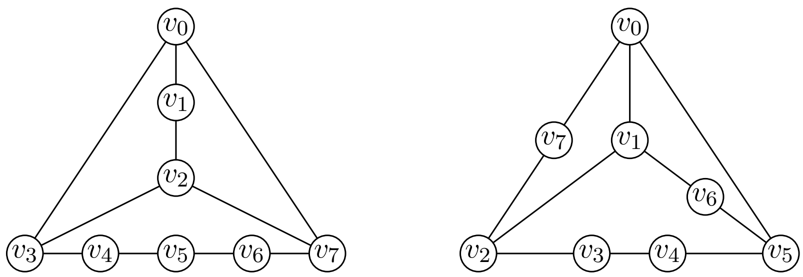

Figure 3: H_3 and H_4

f) **Type VI**:

For $l \geq 9$ and each bipartite subdivision H_l of K_4 of size l, let $D_{l,V(H_l)}$ be the event that the vertices of H_l are properly two-colored in the f-coloring.

From the arguments given above, it follows that if none of the events of the six Types I, II, III, IV, IV and VI described above occurs, then f is a (2,2)-treewidth coloring.

It remains to show that with positive probability none of these events happen. To prove this we apply the Lovász Local Lemma. We construct a dependency graph H whose nodes are all the events of all the six types, in which two nodes X_S and Y_T (where X and Y are one the A, B, C, D events and X and Y respectively depend on the colors of vertices in S and T) are adjacent if and only if $S \cap T \neq \emptyset$.

We need to estimate the number of nodes of each type in H adjacent to any given node. This estimate is given in the following two simple lemmas.

Lemma 3.4 *Let v be an arbitrary vertex of the graph $G = (V, E)$. Then the following four statements hold.*

(i) v belongs to at most d edges of G.

(ii) The number of special pairs containing v is at most $d^{2-\alpha}$.

(iii) For each $t \in \{1, 2\}$, the number of subgraphs of G isomorphic to H_t and containing v is at most $8d^{t+1+3\alpha}$. The number of subgraphs of G isomorphic to H_3 (or H_4) and containing v is at most $8d^{4+3\alpha}$.

(iv) For $l \geq 9$, the number of subgraphs of G on l vertices isomorphic to some bipartite subdivison of K_4 and containing v is at most $\left(l^6 \cdot d^{l-1}\right)/120$.

Proof of Lemma 3.4

Part (i) follows from the fact that $\Delta(G) = d$.

Part (ii) follows from the fact that there are at most d^2 induced paths of length 2 starting from v and for each special pair $\{u, v\}$ there are more than d^α induced paths of length 2 leading to u. Thus the number of special pairs containing v is at most $\frac{d^2}{d^\alpha} = d^{2-\alpha}$.

Proof of Part (iii): Consider the case $t = 3$. There are at most 8 ways of identifying v with a vertex in H_3. Suppose v is identified with v_0. The there are at most d choices each for v_3 and v_7. Once these are fixed there are at most d choices for each of v_4 and v_6. Now there are at most d^α choices for each of v_2 and v_5 since neither v_3 and v_7 nor v_4 and v_6 form a special pair. Now since v_0 and v_2 do not form a special pair, there are at most d^α choices for v_1. Thus there are at most $d^{4+3\alpha}$ ways of embedding H_3 in G so that it contains v in the position of v_0. A similar analysis shows that in each of the other five cases, there are at most $d^{4+3\alpha}$ ways of embedding H_3 in G so that it contains v in a fixed position. This proves (iii) for $t = 3$. The proofs for the cases $t \in \{1, 2, 4\}$ are similar.

Proof of Part (iv) : Note that the number of mutually non-isomorphic bipartite subdivisions of K_4 on l vertices is at most the number of ordered partitions of $l-4$ into six non-negative integers. The latter number is well-known to be $\binom{l+1}{5} \leq l^5/120$. For any such bipartite subdivision H_l, v can be one of the l vertices in H_l. Thus there are at most l ways to fix the position of v in H_l. Since H_l is connected, there is a spanning tree T which is a subgraph of H_l with v as the root and we fix one such spanning tree. Once v is fixed, for each of its neighbors in H_l, i.e. the nodes in the first level in T, there are at most d choices. Similarly, once these node are fixed, the nodes in the next level have at most d choices each. Thus the number of copies of H_l is at most ld^{l-1}. Multiplying this by the number of possible H_ls, we prove Part (iv). ∎

Lemma 3.5 *For $i, j \in \{I, II, III, IV, V, VI\}$, the (i,j)-th entry of the table given below is an upper bound on the number of events of type j which can possibly influence an event of type i.*

	I	II	III	IV	V	$VI(D_{l,V(H_l)})$
I	$2d$	$2d^{2-\alpha}$	$16d^{2+3\alpha}$	$16d^{3+3\alpha}$	$32d^{4+3\alpha}$	$2l^6d^{l-1}/120$
II	$2d$	$2d^{2-\alpha}$	$16d^{2+3\alpha}$	$16d^{3+3\alpha}$	$32d^{4+3\alpha}$	$2l^6d^{l-1}/120$
III	$6d$	$6d^{2-\alpha}$	$48d^{2+3\alpha}$	$48d^{3+3\alpha}$	$96d^{4+3\alpha}$	$6l^6d^{l-1}/120$
IV	$7d$	$7d^{2-\alpha}$	$56d^{2+3\alpha}$	$56d^{3+3\alpha}$	$112d^{4+3\alpha}$	$7l^6d^{l-1}/120$
V	$8d$	$8d^{2-\alpha}$	$64d^{2+3\alpha}$	$64d^{3+3\alpha}$	$128d^{4+3\alpha}$	$8l^6d^{l-1}/120$
$VI(D_{k,V(H_k)})$	kd	$kd^{2-\alpha}$	$8kd^{2+3\alpha}$	$8kd^{3+3\alpha}$	$16kd^{4+3\alpha}$	$kl^6d^{l-1}/120$

Proof of Lemma 3.5

The lemma follows from Lemma 3.4 and the fact that any event is mutually independent of all other events which do not share any vertex with the given event. The upper bound for the number of events of a type Y that can possibly influence an event of type X is obtained by multiplying the number of vertices participating in the event of type X by the upper bound obtained in Lemma 3.4 for the number of events of type Y that contain a given vertex. ∎

Fact 3.6 *(i) For each type I event A, $Pr(A) = \frac{1}{x}$.*
(ii) For each type II event B, $Pr(B) = \frac{1}{x}$.
(iii) For each type III event C, $Pr(C_1) \leq \frac{1}{x^4}$.
(iv) For each type IV event D, $Pr(C_2) \leq \frac{1}{x^5}$.
(v) For each type V event E, $Pr(C_3) \leq \frac{1}{x^6}$.
(vi) For each type VI event D_l,$(l \geq 9)$, $Pr(D_l) \leq \frac{1}{x^{l-2}}$.

We now define the weights y_i to apply the Lemma 2.1.

Recall that c_1 and c_2 are positive constants such that $c_1c_2 = 25$. We choose $c_1 = 6.25$ and $c_2 = 4$.

For an event A of type I, $y_A = \frac{c_2}{x}$. For an event B of type II, $y_B = \frac{c_2}{x}$. For an event of the form C_t, $t \in \{1,2,3\}$, $y_{C_t} = \frac{c_2^{\frac{t+3}{2}}}{x^{t+3}}$. For an event of the form D_l of type VI, $y_{D_l} = \frac{c_2^{\frac{l-2}{2}}}{x^{l-2}}$.

Define

$$T_2 = \left(1 - \frac{c_2}{x}\right);\ T_3 = \left(1 - \frac{{c_2}^2}{x^4}\right);\ T_4 = \left(1 - \frac{{c_2}^{2.5}}{x^5}\right);$$

$$T_5 = \left(1 - \frac{{c_2}^3}{x^6}\right);\ T_6 = \left(1 - \frac{{c_2}^{\frac{l-2}{2}}}{x^{l-2}}\right).$$

By Lemma 2.1, Lemma 3.5 and Fact 3.6, it suffices to verify the following two inequalities, where the first inequality corresponds to events of types I and II and the 2nd inequality to events of types III,IV,V and VI.

$$\frac{1}{x} \quad \leq \quad \frac{c_2}{x} T_2^{2d+2d^{2-\alpha}}\ T_3^{16d^{2+3\alpha}}\ T_4^{16d^{3+3\alpha}}\ T_5^{32d^{4+3\alpha}} \prod_{l\geq 9} T_6^{\frac{2l^6 d^{l-1}}{120}} \quad \ldots (i)$$

For every $k \geq 6$,

$$\frac{1}{x^{k-2}} \quad \leq \quad \frac{{c_2}^{\frac{k-2}{2}}}{x^{k-2}} \cdot {T_2}^{kd+kd^{2-\alpha}} \cdot {T_3}^{8kd^{2+3\alpha}} \cdot {T_4}^{8kd^{3+3\alpha}} \cdot {T_5}^{16kd^{4+3\alpha}} \cdot \prod_{l\geq 9} {T_6}^{\frac{kl^6 d^{l-1}}{120}} \quad \ldots (ii)$$

Simplifying (i), we get:

$${T_2}^{d+d^{2-\alpha}} \cdot {T_3}^{8d^{2+3\alpha}} \cdot {T_4}^{8d^{3+3\alpha}} \cdot {T_5}^{16d^{4+3\alpha}} \cdot \prod_{l\geq 9} {T_6}^{\frac{l^6 d^{l-1}}{120}} \quad \geq \quad \frac{1}{\sqrt{c_2}} \quad \ldots (iii)$$

Simplifying (ii), we get, for $k \geq 6$,

$${T_2}^{d+d^{2-\alpha}} \cdot {T_3}^{8d^{2+3\alpha}} \cdot {T_4}^{8d^{3+3\alpha}} \cdot {T_5}^{16d^{4+3\alpha}} \cdot \prod_{l\geq 9} {T_6}^{\frac{l^6 d^{l-1}}{120}} \quad \geq \quad {c_2}^{\frac{1}{k}-\frac{1}{2}} \quad \ldots (iv)$$

Clearly, proving (iv) for $k = 6$ is sufficient to prove both inequalities (iii) and (iv). We now substitute $c_1 = 6.25\ c_2 = 4$. This yields R.H.S. of (iv) (for $k = 6$) $= (\frac{1}{4})^{\frac{1}{3}}$.

Consider the L.H.S. of (iv) (for $k = 6$). Substituting $x = c_1 c_2 d^{2-\alpha}$ and using the fact that $(1 - \frac{1}{z})^z \geq 1/4$ for all $z \geq 2$, we deduce that L.H.S. of (iv) is at least $(\frac{1}{4})^{S_1+S_2}$, where

$$S_1 \quad = \quad \left(\frac{2}{c_1}\right) + \left(\frac{8}{(c_1\sqrt{c_2})^4 d^{6-7\alpha}}\right) + \left(\frac{8}{(c_1\sqrt{c_2})^5 d^{7-8\alpha}}\right) + \left(\frac{16}{(c_1\sqrt{c_2})^6 d^{8-9\alpha}}\right)$$

$$S_2 \quad = \quad \sum_{l\geq 9} \left(\frac{l^6}{120(c_1\sqrt{c_2})^{l-2} d^{(l-3)-(l-2)\alpha}}\right).$$

Using $\alpha = 6/7$, $c_2 = 4$, $c_1 = 6.25$, $c_1\sqrt{c_2} > 12$ and $2(6.25)^{l-2} \geq l^6$ for $l \geq 9$, we deduce that

$$\begin{aligned} S_1 + S_2 \quad &\leq \quad \frac{2}{c_1} + \frac{24}{(c_1\sqrt{c_2})^4} + \sum_{l\geq 9} \frac{l^6}{120(c_1\sqrt{c_2})^{l-2}} \\ &\leq \quad \frac{2}{6.25} + \frac{2}{12^3} + \sum_{l\geq 9} \frac{1}{60 * 2^{l-2}}. \end{aligned}$$

Thus,

$$S_1 + S_2 \leq \frac{2}{6.25} + \frac{2}{12^3} + \frac{1}{60 * 2^6}$$

which is lesser than $\frac{1}{3}$. Hence inequality (4) is proved.

Thus by Lovász Local Lemma, with probability greater than zero none of the bad events occurs and hence there exists a $(2, 2)$-treewidth coloring using $25d^{\frac{8}{7}}$ colors. This completes the proof of Proposition 3.2 and hence of Theorem 1.8. ∎

4 Lower bounds

We first prove that Theorem 1.9 is a consequence of Theorem 1.5 and then prove the latter result. To do this we need a characterization of treewidth due to Seymour and Thomas [18].

Definition 4.1 *Let $G = (V, E)$ be a graph. Two subsets $W_1, W_2 \subset V$ are said to* touch *if they have a vertex in common or if there is an edge $(w_1, w_2) \in E$ such that $w_1 \in W_1$, $w_2 \in W_2$. A set B of mutually touching connected vertex sets is called a* bramble. *A* hitting set *for B is a set which intersects every element of B. The* order *of a bramble B is the size of a minimum hitting set for B. The* bramble number *of G is the maximum order of all brambles of G.*

Theorem 4.2 *(Seymour and Thomas [18]) Let k be a non-negative integer. A graph has treewidth k if and only if it has bramble number $k + 1$.*

Corollary 4.3 *If G has a bramble of order k, $tw(G) \geq k - 1$.*

Proof of Theorem 1.9 Observe that any $(2, k)$-treewidth coloring is also a $(2, H)$-subgraph coloring for any bipartite graph H of treewidth more than k. Thus by Theorem 1.5, it suffices to prove that there exists a bipartite graph H having treewidth greater than k and having $(k^2 + 5k + 2)/2$ edges.

Consider the bipartite graph $H = (V, E)$ where

$$V = \{a_1, a_2, ..., a_{k+1}\} \cup \{b_1, b_2, ..., b_{k+1}\} \text{ and}$$

$$E = \{\ (a_i, b_j) : 1 \leq i \leq j \leq k+1\ \} \bigcup \{\ (a_i, b_1) : 2 \leq i \leq k+1\ \}.$$

The number of edges in this graph is $\binom{k+1}{2} + 2k + 1 = (k^2 + 5k + 2)/2$.

Consider the following bramble B in H.

$$B = \{\{a_1\}, \{b_1\}\} \bigcup \{\ \{a_i, b_i\} : 2 \leq i \leq k+1\ \}.$$

It is clear that any hitting set of B has to have size at least $k + 2$. Hence by Corollary 4.3, $tw(H) \geq k + 1$.

Proof of Theorem 1.5

The proof is based on analyzing a random graph $G(n, p)$ for a suitably chosen value of p and is similar to the approach used by Alon, McDiarmid and Reed [1].

Let $V = \{1, 2, ..., n\}$ be a set of n labelled vertices.

Choose $p = c\left(\frac{\log n}{n}\right)^{\frac{1}{m}}$, where $c > 0$ is a constant, independent of n, to be chosen later, and let $G \in G_{n,p}$, $G = (V, E)$ be a random graph on V obtained by choosing each pair of distinct members of V independently to be an edge with probability p. Let d be the maximum degree of G. By known results about the degrees of random graphs, (see for example, [5]), we have for μ defined by $\mu = (n-1)p \approx np = cn^{1-\frac{1}{m}}(\log n)^{\frac{1}{m}}$,

$$\mathbf{Pr}\left(\mu/2 \leq d \leq 2\mu\right) \to 1 \text{ as } n \to \infty \tag{3}$$

Let H be the bipartite graph in Theorem 1.5 and $V(H) = X \cup Y$ be a bipartition into independent sets X and Y such that $r = max\{|X|, |Y|\}$.

We first claim that for any fixed partition of $V = V(G)$ into $s \leq n/r$ disjoint parts, the probability that this partition is a $(2, \{H\})$-coloring of G is at most $(1-p^m)^{\binom{n/r^2}{2}}$.

Let $V_1, ..., V_s$ be the parts of the partition. For each V_i, remove at most $r-1$ smallest (with respect to some fixed linear ordering of V) vertices to obtain a V_i' such that $|V_i'| \equiv 0 \pmod r$. The number of removed vertices is at most $s(r-1) \leq n(r-1)/r$ so that the graph induced by the union of the V_i's has at least n/r vertices. Now partition each V_i' into subsets of size r so that we get at least $\lceil n/r \rceil$ vertices partitioned into subsets $U_1, U_2, ..., U_k$ of cardinality r each, where $k \geq n/r^2$. For every i, j such that $1 \leq i < j \leq k$, the probability that the bipartite subgraph formed by $U_i \bigcup U_j$ does not contain a copy of H is at most $1 - p^m$. Since all these $\binom{k}{2}$ events are mutually independent, the probability that for every $i < j$, $U_i \cup U_j$ has no copy of H is at most $(1-p^m)^{\binom{n/r^2}{2}}$ and this probability is an upper bound on the required probability thereby proving the claim in the preceding paragraph.

The total number of partitions of V is at most n^n. Hence the probablity that there exists a partition $V = V_1 \cup \ldots \cup V_s$ $(s \leq n/r)$ which forms a $(2, H)$-subgraph coloring is at most

$$n^n(1-p^m)^{\binom{n/r^2}{2}}$$
$$< exp\left(n\log n - \binom{n/r^2}{2}p^m\right)$$

Since $p = c(\log n/n)^{\frac{1}{m}}$, we choose c such that $c^m > 2r^4$, so that this probability is $o(1)$.

Therefore, $Pr[\chi_{2,\{H\}}(G) > n/r] \to 1$ as $n \to \infty$.

Combining this with (3), we see that there exist graphs G such that $d = \Delta(G) \leq 2cn^{1-\frac{1}{m}}(\log n)^{\frac{1}{m}}$ and $\chi_{2,\{H\}}(G) > n/r$. Hence, $\chi_{2,\{H\}}(G) = \Omega\left(\frac{d^{1+\frac{1}{m-1}}}{(\log n)^{\frac{1}{m-1}}}\right) = \Omega\left(\frac{d^{1+\frac{1}{m-1}}}{(\log d)^{\frac{1}{m-1}}}\right)$ using $\log d = \Omega(\log n)$.

The proof of the bounds for $\chi_{2,\{H\}}(d, bip)$ is a slight modification of the above argument and is based on considering a random bipartite graph $G \in G(n,n,p)$ obtained by including each of the n^2 edges independently with probability p between two independent sets of size n each. This completes the proof of Theorem 1.5. ∎

5 Extensions to colorings with several families forbidden simultaneously

It is also possible to extend our results to more restricted colorings where we require simultaneously for several pairs $(j_i, \mathcal{F}_i)$ $(i = 1, \ldots, l)$ that the union of *any* j_i color classes has no copy of any member of $\mathcal{F}_i$. Such colorings are precisely the kind of colorings considered by Nešetřil and Ossona de Mendez in [15] for families of H-minor-free graphs. This notion generalizes the kind of colorings studied by DeVos, et. al. in [7] for families of H-minor-free graphs. See also [20] for some related work on some similar colorings by Zhu. However, we obtain bounds which work for any arbitrary graph G. We first formally define these colorings.

Definition 5.1 *Let $\mathcal{P} = \{(j_1, \mathcal{F}_1), \ldots, (j_l, \mathcal{F}_l)\}$ be a set of l pairs such that for each $i \leq l$, j_i is a positive integer and $\mathcal{F}_i$ is a family of connected graphs of (usual) chromatic number at most j_i such that for each $H \in \mathcal{F}_i$, $|V(H)| > j_i$. We define a $\mathcal{P}$-subgraph coloring to be a proper coloring of $V(G)$ so that, for each $i \leq l$, the subgraph of G induced by the union of any j_i color classes does not contain an isomorphic copy of H as a subgraph, for each $H \in \mathcal{F}_i$. We denote by $\chi_{\mathcal{P}}(G)$ the minimum number of colors used by any $\mathcal{P}$-subgraph coloring of G.*

As before we shall first consider colorings in which we restrict the size of the maximum connected component in the union of color classes and then derive, as a consequence, bounds for the type of coloring defined above.

Definition 5.2 *Let $\mathcal{T} = \{(j_1, k_1), \ldots, (j_l, k_l)\}$ where the j_i's and k_i's are positive integers such that $j_i \leq k_i$ for each $i \in \{1, \ldots, l\}$. We define a $\mathcal{T}$-coloring of a graph G to be a proper coloring of $V(G)$ so that in the union of any j_i color classes, each connected component has size at most k_i for each $i \in \{1, \ldots, l\}$. We denote by $\chi_{\mathcal{T}}^{con}(G)$ the minimum number of colors used by any $\mathcal{T}$-coloring of $V(G)$.*

We now present the main results of this section.

Theorem 5.3 *Let $\mathcal{T} = \{(j_1, k_1), \ldots, (j_l, k_l)\}$ be as defined before. Then there exists a constant $C = C(\mathcal{T})$ such that for any graph G of maximum degree d, $\chi_{\mathcal{T}}^{con}(G) \leq C d^{max_i \frac{k_i}{k_i+1-j_i}}$ where we choose*

$$C = C(\mathcal{T}) = \max_i \left(4l(k_i+1)(12j_i)^{k_i+1}\right)^{\frac{1}{k_i+1-j_i}}.$$

We skip the proof of the above theorem as it is based on an application of the Lovász Local Lemma and is a generalization of the proof of Theorem 1.3. The above theorem immediately leads to an upper bound for $\mathcal{P}$-subgraph colorings.

Corollary 5.4 *Let $\mathcal{P} = \{(j_1, \mathcal{F}_1), \ldots, (j_l, \mathcal{F}_l)\}$ be as defined in Definition 5.1. For each i, let k_i (with $k_i > j_i$) denote the size of the smallest graph in $\mathcal{F}_i$. Then there exists a constant $C = C((j_1, k_1), \ldots, (j_l, k_l))$ such that for any graph G of maximum degree d, $\chi_{\mathcal{P}}(G) \leq Cd^{max_i \frac{k_i - 1}{k_i - j_i}}$.*

By setting $\mathcal{P}_l = \{(1, \mathcal{F}_1), \ldots, (l, \mathcal{F}_l)\}$ where $\mathcal{F}_i$ is the set of all i-colorable (usual coloring) graphs of treewidth i, for each $i \leq l$, we can get upper bounds on the the type of colorings studied by DeVos, et. al. in [7]. The proof of this result follows essentially from the proof arguments of Part (i) of Theorem 1.8 (on low treewidth colorings).

Corollary 5.5 *For $l \geq 1$, let $\chi_{\mathcal{P}_l}(G)$ denote the minimum number of colors sufficient to obtain a proper coloring of $V(G)$ so that the union of any $j \leq l$ color classes forms a subgraph of treewidth at most $j - 1$. Then, there exists a constant $C = C(l)$ such that for any graph of maximum degree d, $\chi_{\mathcal{P}_l}(G) \leq Cd^{l-1+(1/l)}$.*

6 Improved bounds on $\chi_{j,k}^{con}(G)$

A further improvement of the bound obtained in Theorem 1.3 is possible. For a graph G, let $X = X_k(G)$ denote the maximum number of connected induced subgraphs of size $k+1$ containing any given vertex. We noted that d^k is an asymptotic upper bound (treating k a constant) on $X_k(G)$ and which works for any graph of maximum degree d. But, the actual value of $X_k(G)$ could be far smaller for several classes of graphs. For example, in the case of cycles, this number is exactly $k+1$. The new bound is stated in terms of $X_k(G)$ and is stated in the following theorem.

Theorem 6.1 *Let $j \geq 2$, k be given positive integers such that $j \leq k$. Then, for any graph G with $X = X_k(G)$, we have $\chi_{j,k}^{con}(G) \leq CX^{\frac{1}{k+1-j}}$ where $C = C(j, k) = k\left(2\,(3j)^{k+1}\right)^{\frac{1}{k+1-j}}$.*

Proof We first obtain a coloring which is the same as a (j, k)-coloring except that it does not have to obey the properness requirement. Later in the proof, we show that such a coloring exists which uses $S = DX^{\frac{1}{k+1-j}}$ colors where $D = C/k$. Let $V_1, \ldots, V_S$ be the color classes. Consider any j. Since $G[V_j]$ has maximum degree at most $k-1$, we can obtain proper k-coloring of G_j. We replace each V_j by its respective k-coloring. It is easy to see that the resulting coloring is a (j, k)-coloring using the permitted number of colors.

The proof of the existence of a (j, k)-coloring which need not be proper, is essentially the same as the proof of Theorem 1.3 except that we now redefine x to be $x = (C/k)X^{\frac{1}{k+1-j}}$. Also, we need to ensure the absence of only events of Type II. From the definition of X, it follows that any such event is independent of all but at most $(k+1)X$ other events. Thus, it suffices to only ensure the following inequality.

$$\frac{j^{k+1}}{x^{k+1-j}} \leq \frac{(3j)^{k+1}}{x^{k+1-j}} \left(1 - \frac{(3j)^{k+1}}{x^{k+1-j}}\right)^{(k+1)X} \tag{4}$$

which is equivalent to proving the inequality

$$\frac{1}{3} \leq \left(1 - \frac{(3j)^{k+1}}{x^{k+1-j}}\right)^{X} \tag{5}$$

This follows from proving $\frac{1}{3} \leq 4^{-1/2}$ which is always true. ∎

This leads to improved estimates on other chromatic numbers in terms of $X_k(G)$ for suitable k.

7 Conclusions and Some Remarks

We proved an upper bound of $O(d^{\frac{k}{k+1-j}})$ on the (j, k)-chromatic number of graphs of maximum degree d and used it to obtain upper bounds for forbidden subgraph colorings. But forbidding all connected graphs on certain number of vertices is a stronger requirement than what is expected and does not make use of the structure of the members of the forbidden family and so there is scope for further improving the upper bounds for each of these three types of colorings for several *specific* families of forbidden graphs. Theorem 6.1 is a beginning in this direction.

In particular, in a recent work [4], the present authors obtained the upper bound of $Cd^{\frac{m}{m-1}}$ on the value of $\chi_{2,\mathcal{F}}(d)$. Here, C depends only on $\mathcal{F}$ and m has the meaning stated in Theorem 1.5. In view of this theorem, the above-mentioned upper bound is tight within a $(\log d)^{1/(m-1)}$ multiplicative factor.

Theorem 1.5, Corollary 1.6 and Theorem 1.9 provide lower bounds on the maximum value (for a given d) of the respective chromatic numbers for the case $j = 2$. It is likely that similar lower bounds can be obtained for each $j > 2$ and it is worth exploring in this direction.

The *degeneracy* of a graph is the maximum value of the minimum degree of any induced subgraph. It is known that that forests are precisely the graphs having degeneracy at most one. In particular, degeneracy of $K_{1,n}$ is exactly one. But, it is easy to verify that $\chi_{2,k}^{con}(K_{1,n}) \geq n/(k-1)$. This shows that $(2, k)$-chromatic number cannot be bounded by a function of degeneracy alone.

Let SK_n denote the fully subdivided K_n. That is, subdivide each edge of K_n by replacing it with the addition of a new vertex which is adjacent to both endpoints. SK_n has maximum degree $n-1$ and its degeneracy is 2. It can be verified that for any k such that $n \geq 3k!$, every proper coloring of SK_n using k colors is not acyclic. Hence, $a(SK_n) = \Omega((\log n)/(\log\log n))$. This shows that $a(G)$ cannot be bounded by a function of degeneracy alone.

Suppose we only require that the subgraph induced by the union of any j color classes has no induced copy of any member of $\mathcal{F}$. Every $(j, \mathcal{F})$-subgraph coloring obeys this requirement and hence the bound of Corollary 1.4 applies to such colorings also.

One can also define proper edge colorings with similar restrictions on the union of any few color classes, generalizing some well-known colorings like acyclic edge colorings. Some recent results on tight upper bounds on such edge colorings will appear in a forthcoming paper [3].

Also, one can look at algorithmic issues associated with these coloring notions. Note that the problem of testing whether $\chi_{j,\mathcal{F}}(G) \leq k$ for an input graph G and input parameter k is NP-complete even for some specific $(j, \mathcal{F})$ (examples : $(1, \mathcal{F}_1)$, $(2, \mathcal{F}_2)$ where $\mathcal{F}_1 = \{K_2\}$ and $\mathcal{F}_2$ is the set of cycles). It would be interesting to determine the computational complexity of this problem for other specific pairs $(j, \mathcal{F})$.

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