Forbidden Subgraph Colorings and the Oriented Chromatic Number

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Abstract. We present an improved upper bound of $O(d^{1+\frac{1}{m-1}})$ for the $(2,\mathcal{F})$-subgraph chromatic number $\chi_{2,\mathcal{F}}(G)$ of any graph G of maximum degree d. Here, m denotes the minimum number of edges in any member of $\mathcal{F}$. This bound is tight up to a $(\log d)^{1/(m-1)}$ multiplicative factor and improves the previous bound presented in [1].

We also obtain a relationship connecting the oriented chromatic number $\chi_o(G)$ of graphs and the $(j,\mathcal{F})$-subgraph chromatic numbers $\chi_{j,\mathcal{F}}(G)$ introduced and studied in [1]. In particular, we relate oriented chromatic number and the $(2,r)$-treewidth chromatic number and show that $\chi_o(G) \leq k\left((r+1)2^r\right)^{k-1}$ for any graph G having $(2,r)$-treewidth chromatic number at most k. The latter parameter is the least number of colors in any proper vertex coloring which is such that the subgraph induced by the union of any two color classes has treewidth at most r.

We also generalize a result of Alon, et. al. [2] on acyclic chromatic number of graphs on surfaces to $(2,\mathcal{F})$-subgraph chromatic numbers and prove that $\chi_{2,\mathcal{F}}(G) = O(\gamma^{m/(2m-1)})$ for some constant m depending only on $\mathcal{F}$. We also show that this bound is nearly tight. We then use this result to show that graphs of genus g have oriented chromatic number at most $2^{O(g^{1/2+\epsilon})}$ for every fixed $\epsilon > 0$. This improves the previously known bound of $2^{O(g^{4/7})}$. We also refine the proof of a bound on $\chi_o(G)$ obtained by Kostochka, et. al. in [3] to obtain an improved bound on $\chi_o(G)$.

1 Introduction

We study several variants of proper vertex colorings and present relationships connecting them. The chromatic number $\chi(G)$ of G is the least k such that G is properly colorable using k colors. An acyclic vertex coloring (introduced in [4], see also [5]) of $G = (V,E)$ is a proper coloring of V in which the subgraph induced by the union of any two color classes is acyclic. The acyclic chromatic number $a(G)$ is the least k such that G admits an acyclic vertex coloring using k colors.

Sopena, in ([6]), introduced the notion of oriented chromatic number for oriented graphs (directed graphs having no self-loops and no 2-cycles). The oriented chromatic number of an oriented graph $\overrightarrow{G}$ is the smallest size $|V(\overrightarrow{H})|$ of a $\overrightarrow{H}$ for

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which a homomorphism $\phi : \overrightarrow{G} \to \overrightarrow{H}$ exists. Equivalently, $\chi_o(\overrightarrow{G})$ is the smallest $k \geq 1$ such that there is a proper k-coloring $(V_1, \ldots, V_k)$ of $V(\overrightarrow{G})$ such that for every $i \neq j$, all edges joining V_i and V_j are oriented in the same way.

The oriented chromatic number $\chi_o(G)$ of an undirected graph G is the maximum value of $\chi_o(\overrightarrow{G})$ where the maximum is over all orientations $\overrightarrow{G}$ of G. Bounds for the oriented chromatic number have been obtained in terms of the maximum degree and also for special families of graphs such as trees, planar graphs, partial k-trees([6]), etc. Of these, the following two results are relevant to the main results of this paper. They are:

(B1) The result of Sopena in [6] that, for every $r \geq 1$, every partial r-tree has oriented chromatic number at most $(r+1)2^r$.
(B2) The result of Raspaud and Sopena in [7] that if a graph has acyclic chromatic number at most k, then $\chi_o(G) \leq k2^{k-1}$.

In this paper, we obtain improved bounds on $(2, \mathcal{F})$-subgraph chromatic numbers (of which acyclic chromatic number is a special case) and also obtain a relation involving these numbers and oriented chromatic numbers. We then apply these results to obtain new or improved bounds on $(2, \mathcal{F})$-subgraph chromatic numbers and oriented chromatic numbers of graphs of bounded genus.

1.1 Improved Bounds on $(2, \mathcal{F})$-Chromatic Numbers

Recently, the present authors [1] studied a generalized notion of proper colorings which impose constraints on the union of any few color classes. Such a notion was first considered by Nesetril and Ossona de Mendez in [8], where it is proved that these numbers are bounded for proper minor-closed families of graphs. For suitably chosen constraints, this general notion specializes to known restricted colorings like acyclic colorings, star colorings, etc. We need the following definitions from [1], which formally define a general restricted coloring.

Given two graphs G and H, we say that G is H-free if G has no isomorphic copy of H as a subgraph (not necessarily induced). Given a family $\mathcal{F}$ of graphs, we say that G is $\mathcal{F}$-free if G is H-free for each $H \in \mathcal{F}$.

Definition 1. *Let j be a positive integer and $\mathcal{F}$ be a family of connected graphs of (usual) chromatic number at most j such that for each $H \in \mathcal{F}$, $|V(H)| > j$. We define a $(j, \mathcal{F})$-subgraph coloring to be a proper coloring of the vertices of a graph G so that the subgraph of G induced by the union of any j color classes is $\mathcal{F}$-free. We denote by $\chi_{j,\mathcal{F}}(G)$ the minimum number of colors sufficient to obtain a $(j, \mathcal{F})$-subgraph coloring of G.*

In [1], we obtained the bound of $\chi_{j,\mathcal{F}}(G) = O\left(d^{\frac{k-1}{k-j}}\right)$ for any graph G of maximum degree d, where k is $\min_{H \in \mathcal{F}} |V(H)|$. But this bound is not optimal; for acyclic coloring, we have $j = 2$, $\mathcal{F} = \{C_4, C_6, \ldots\}$ and $k = 4$ and hence we get a bound of $O(d^{3/2})$ but it is known [9] that $a(G) = O(d^{4/3})$. Our first main result is the following improved bound on the $(2, \mathcal{F})$-chromatic numbers. This improves the above bound of [1] for the case $j = 2$.

Theorem 1. *Let $\mathcal{F}$ be a family of connected bipartite graphs on 3 or more vertices such that the minimum number of edges in any member of $\mathcal{F}$ is m. Then, for any graph G of maximum degree d, $\chi_{2,\mathcal{F}}(G) < \lceil Cd^{1+\frac{1}{m-1}} \rceil$ where $C = C(\mathcal{F}) = 64(m+1)^3 s$ and s is the number of bipartite graphs in $\mathcal{F}$ on at most m vertices.*

For acyclic vertex coloring, we note that this leads to the optimal bound of $O(d^{4/3})$ since $\mathcal{F} = \{C_4, C_6, \ldots\}$ so that $m = 4$ in this case. In fact, for every constant family $\mathcal{F}$, the upper bound of Theorem 1 is tight within a multiplicative factor of $O((\log d)^{1/(m-1)})$; this follows from the results of [1].

1.2 Relating $\chi_{j,\mathcal{F}}(G)$ and $\chi_o(G)$

Our second main result is the following connection between $(j, \mathcal{F})$-subgraph colorings and oriented colorings. This generalizes and was inspired by the connection **(B2)** between $a(G)$ and $\chi_o(G)$ established in [7]. For a family $\mathcal{F}$ of connected graphs, let $Forb(\mathcal{F}) = \{G : G \text{ is } \mathcal{F} - \text{free}\}$.

Theorem 2. *Let $\mathcal{F}$ be a family of connected graphs. Suppose there exists a natural number t such that $\chi_o(F) \le t$, for each $F \in Forb(\mathcal{F})$. Suppose $j \ge 2$. Then, for any graph $G \notin Forb(\mathcal{F})$ with $\chi_{j,\mathcal{F}}(G) \le k$, its oriented chromatic number $\chi_o(G)$ is at most $kt^{\lceil \frac{2k-j}{j} \rceil}$ if j is even and is at most $kt^{\lceil \frac{2k-j+1}{j-1} \rceil}$ if j is odd.*

In Section 2, we prove this theorem. The special case of this theorem obtained by setting $j = 2$, is going to be used later and we state it separately as the following theorem.

Theorem 3. *Let $\mathcal{F}$ be a family of connected graphs. Suppose there exists a t such that $\chi_o(F) \le t$, for each $F \in Forb(\mathcal{F})$. Then, for any graph $G \notin Forb(\mathcal{F})$ with $\chi_{2,\mathcal{F}}(G) \le k$, its oriented chromatic number $\chi_o(G)$ is at most kt^{k-1}.*

Fixing $\mathcal{F} = \{G : \chi(G) \le j, \ tw(G) \ge r+1\}$, Definition 1 specializes to the following:

Definition 2. *Let j, r be positive integers such that $j \le r+1$. We define a (j, r)-treewidth (vertex) coloring of a graph $G = (V, E)$ to be a proper coloring of $V(G)$ so that the subgraph induced by the union of any j color classes has treewidth at most r. We denote by $\chi^{tw}_{j,r}(G)$ the minimum number of colors sufficient to obtain a (j, r)-treewidth coloring of G.*

We now specialize Theorem 3 by choosing $\mathcal{F}$ to be the set of all connected bipartite graphs of treewidth $r+1$ and apply the bound **(B1)** on the oriented chromatic number of partial r-trees to obtain the following result which we shall later use to bound the oriented chromatic number of graphs on surfaces.

Corollary 1. *For $r \ge 2$, let G be any graph with a $(2, r)$-treewidth chromatic number at most k. Then, $\chi_o(G)$ is at most $k((r+1)2^r)^{k-1}$.*

1.3 $(2, \mathcal{F})$-Subgraph Colorings of Graphs on Surfaces

It is known from the Map Color Theorem of Ringel and Youngs [10] that the chromatic number of an arbitrary surface of Euler characteristic $-\gamma$ is $\Theta(\gamma^{1/2})$. Using the $O(d^{4/3})$ bound (for general graphs), Alon, Mohar and Sanders proved in [2] that the acyclic chromatic number of a (simple) graph embeddable on a surface of characteristic $-\gamma(\leq 0)$ is at most $100\gamma^{\frac{4}{7}} + 10^4$. It was also shown that this bound is nearly tight.

Generalizing these arguments and by using the bound of Theorem 1, we prove that this result can be extended for $(2, \mathcal{F})$-colorings as well provided that $\mathcal{F}$ does not contain connected graphs with pendant vertices. Our third main result is this extension. which we state below.

Theorem 4. *Let $\mathcal{F}$ be a family of connected bipartite graphs on at least 4 vertices each having minimum degree at least 2. Let m be the smallest number of edges of any member of $\mathcal{F}$. If G is a (simple) graph embeddable on a surface of Euler characteristic $-\gamma \leq 0$, then $\chi_{2,\mathcal{F}}(G) \leq A\gamma^{\frac{m}{2m-1}} + B$ where A and B are constants depending only on $\mathcal{F}$.*

For the acyclic chromatic number, we have $m = 4$ and $m/(2m-1) = 4/7$; this is the bound obtained in [2]. By choosing $\mathcal{F} = \mathcal{F}_r$ where $\mathcal{F}_r$ is the set of all connected bipartite graphs of treewidth $r+1$, we get the following consequence of Theorem 4.

Corollary 2. *If G is a simple graph embeddable on a surface of Euler characteristic $-\gamma \leq 0$, then, $\chi^{tw}_{2,r}(G) \leq A\gamma^{\frac{m_r}{2m_r-1}} + B$ for every $r \geq 1$. Here, A and B are suitable absolute positive constants and m_r denotes the minimum number of edges in any member of $\mathcal{F}_r$.*

We also establish that the upper bound of Theorem 4 is tight upto a $polylog(\gamma)$ multiplicative factor. This generalizes a similar tightness result presented in [2] for acyclic chromatic numbers.

Theorem 5. *Let $\mathcal{F}$ and m be as described in Theorem 4. For every sufficiently large $\gamma \geq 0$, there is a graph G embeddable on a surface (orientable or non-orientable) with Euler characteristic $-\gamma$: $\chi_{2,\mathcal{F}}(G) \geq c\gamma^{\frac{m}{2m-1}}/(\log\gamma)^{1/(2m-1)}$, for some positive constant c which depends only on $\mathcal{F}$.*

1.4 Oriented Chromatic Numbers on Surfaces

For graphs of Euler characteristic $-\gamma \leq 0$, by combining the bound $a(G) = O(\gamma^{4/7})$ obtained in [2] with the bound **(B2)** of [7] (mentioned before), we get an upper bound of $O(\gamma^{4/7}2^{O(\gamma^{4/7})}) = 2^{O(\gamma^{4/7})}$ for the oriented chromatic number $\chi_o(G)$. The fourth main result of this paper is an improvement of this bound and is obtained by combining Corollary 1 and Corollary 2. Recall that Corollary 1 is a generalization of bound **(B2)** and Corollary 2 is a generalization of the bound obtained in [2].

Theorem 6. *Let G be a simple graph embeddable on a surface of Euler characteristic $-\gamma \leq 0$. Then,*

$$\chi_o(G) \leq O(\gamma^{\frac{m_r}{2m_r-1}})((r+1)2^r)^{O(\gamma^{\frac{m_r}{2m_r-1}})} \leq 2^{O(\gamma^{m_r/(2m_r-1)})}$$

for every fixed $r \geq 1$. Thus for every $\epsilon > 0$, there exists c_ϵ such that $\chi_o(G) \leq 2^{c_\epsilon \gamma^{(1/2)+\epsilon}}$.

Proof: Follows as a consequence of combining Corollary 1 and Theorem 4 with the bound **(B1)** (mentioned earlier).

Note that this significantly improves the bound $2^{O(\gamma^{4/7})}$ mentioned before.

1.5 Outline of the Paper

We prove Theorem 1 in Section 2. Theorem 2 is proved in Section 3. Theorem 4 and Theorem 5 are proved in Section 4. In Section 5, we refine the proof of a bound on $\chi_o(G)$ (in terms of its maximum degree) obtained by Kostochka, Sopena and Zhu in [3] to obtain an improved bound on $\chi_o(G)$. Finally, in Section 6, we conclude with remarks on some related issues.

2 Proof of Theorem 1

The proof is based on probabilistic arguments. The probabilistic tool we use is an easy-to-use version of the Lovasz Local Lemma, which we state below without proof (will appear in the journal version).

Lemma 1. *(Special case of Lovász Local Lemma) Consider a finite set of events which can be partitioned into types 1,2,... such that the probability of any event of type i is at most p_i and let the events of type i be $A_{i,1}, A_{i,2}, \ldots$. Further, let there be reals $a_1, a_2, \ldots$ and $b_1, b_2, \ldots$ (each $b_i \geq 1$) such that any event of type i is independent of all but at most $a_i * b_j$ events of type j. Suppose, also that* **(A) :** $\sum_i 2^{(a_i+1)} b_i p_i \leq 1$ *holds. Then,*

$$Pr(\cap(\overline{A_{i,j}})) > 0$$

i.e. with positive probability none of the events $A_{i,j}$ holds. In particular, if the number of types of events is k and $k2^{a_i+1} b_i p_i \leq 1$ for each $i \in [k]$, then with positive probability, none of the events $A_{i,j}$ hold.

Proof of Theorem 1:
Choose $x = \lceil Cd^{1+\beta} \rceil$ where $\beta = \frac{1}{m-1}$ and $C = C(\mathcal{F}) = 64(m+1)^3 s$.

Let $f : V \rightarrow \{1, 2, ..., x\}$ be a random vertex coloring of G, where for each vertex $v \in V$ independently, the color $f(v) \in \{1, 2, ..., x\}$ is chosen uniformly at random. It suffices to prove that with positive probability, f is a $(2, \mathcal{F})$-coloring of G. To this end, we define a family of bad events whose total failure implies a $(2, \mathcal{F})$-coloring and use the Lovasz local Lemma (as stated in Lemma 1) to show that with positive probability none of them occurs. The events we consider are of the following types.

a) **Type 1:** For each pair of adjacent vertices u and v, let $A_{u,v}$ be the event that $f(u) = f(v)$.

To reduce the number of copies of forbidden subgraphs we need to consider, we use a notion similar to the one employed in [9]. A subset of k vertices is called a special k-set if there are more than $d^{1-(k-1)\beta}$ vertices adjacent to each of the k vertices.

We say that a subset S of the vertices is *good* if for every vertex $v \in S$ and for any $k \in [2, m]$, the set of neighbors of v in S does not contain any special k-set as a subset.

For each $k \in [2, m]$, we define the following events:

b) **Types 2,k:** For each special set S of k vertices, let $B_k(S)$ be the event that the vertices of S are colored with one common color by f.

c) **Type 3:** For each connected subset L of $V(G)$ such that $|L| = m + 1$, let C_L be the event that the vertices in L are colored using at most 2 colors in the coloring by f.

Let the bipartite members of $\mathcal{F}$ of size at most m be H_1, H_2,...,H_s where $s = s(\mathcal{F})$ is the number of such members. For each $i \in [1, s]$, we define the following Type $4, i$ events:

d) **Type 4,i:** For each good subset S of vertices of G such that $G[S]$ contains H_i as a spanning subgraph, let $D_i(S)$ be the event that the random coloring f uses at most 2 colors on the vertices of S.

If we forbid all events of Types 1 and $(2, k)$, then for any $S \subseteq V$ such that (i) $G[S]$ contains some H_i as a spanning subgraph and (ii) S is not a good set, there should be some $v \in S$ and some $k \in [2, m]$ such that $N_S(v)$ contains a special k-set which is not monochromatically colored (since events of Type 2,k are forbidden) and hence f uses at least 3 colors on S.

Thus, it follows that if none of the events of the above types occur, then f is a $(2, \mathcal{F})$-coloring. We first estimate upper bounds on the probabilities of each type of events.

(i) For each Type 1 event A, $p_1 = Pr(A) = \frac{1}{x}$.
(ii) For each Type $(2, k)$ event B_k, $p_{2,k} = Pr(B_k) = \frac{1}{x^{k-1}}$.
(iii) For each Type 3 event C, $p_3 = Pr(C) \leq \frac{1}{x^{m-1}}$.
(iv) For each Type $(4, i)$ event D_i, $p_{4,i} = Pr(D_i) \leq \frac{2^{n_i}}{x^{n_i-2}}$.

Note that any of the events defined above is mutually independent of all events that do not share a vertex in common with the given event. Thus, it suffices to estimate the number of events of each type containing a given vertex. This estimate is given in the following simple lemma.

Claim 2
Let v be an arbitrary vertex of the graph $G = (V, E)$. Then the following two statements hold.

(i) v belongs to at most d edges of G.

(ii) For each $k \in [2, m]$, the number of special k-sets containing v is at most $d^{(k-1)(1+\beta)}$.

(iii) v belongs to at most $(m+1)4^{m+1}d^m$ connected subsets of size $m+1$ in $V(G)$.

(iv) For each $i \in [1, m+1]$, v belongs to at most $n_i d^{(n_i-2)(1+\beta)}$ subgraphs isomorphic to H_i where $n_i = |V(H_i)|$ and such that the vertex set of the subgraph is *good*.

Proof of Claim 2

Part (i) follows from the fact that $\Delta(G) = d$.

Part (ii) follows from the fact that there are at most d^k induced stars of size $k+1$ in G, with v as a leaf, and for each special k-set there are more than $d^{1-(k-1)\beta}$ centers of the $k+1$-star. Thus the number of special k-sets containing v is at most $\frac{d^k}{d^{1-(k-1)\beta}} = d^{(k-1)(1+\beta)}$.

Part (iii) has already been established as part of the proof of Proposition 2.2 in [1].

Part (iv) can be seen as follows: The position of v in H_i has atmost n_i choices. Once v is identified with a vertex of H_i, the number of ways of embedding the remaining vertices can be bounded as follows: consider a sequence $v_2, ..., v_{n_i}$ of the remaining vertices of H_i such that each vertex has atleast one neighbour to its left in the sequence. Clearly this is possible since H_i is connected. Let t_i denote the number of vertices to the left of v_i and adjacent to it. Once the vertices to the left of v_i are embedded in G, the number of ways of identifying v_i in G is at most $d^{1-(t_i-1)\beta}$ because there is no special t_i set among these vertices. Thus the number of ways of embedding the remaining vertices of H_i in G is at most $d^{\sum_{i=2}^{n_i}[1-(t_i-1)\beta]}$. Using the fact that $\sum_{i=2}^{t_i} = |E(H_i)| \geq m$ and $\beta = \frac{1}{m-1}$, we see that $\sum_{i=2}^{n_i}[1-(t_i-1)\beta] \leq (n_i-1)(1+\beta) - m\beta = (n_i-2)(1+\beta)$. This proves Part (iv) and completes the proof of Claim 2.

Since an event is independent of all other events with which it does not share a vertex, we see that the assumptions of Lemma 1 hold with the following values of a_is and b_is.

Type 1 : $a_1 = 2$, $b_1 = d$.
Type 2,k : $a_{2,k} = k$, $b_{2,k} = d^{(k-1)(1+\beta)}$ for each $k \in [2, m]$.
Type 3 : $a_3 = m+1$, $b_3 = (m+1)4^{m+1}d^m$.
Type 4,i : $a_{4,i} = n_i$, $b_{4,i} = n_i d^{(n_i-2)(1+\beta)}$ for each $i \in [1, s]$.

By Lemma 1, to prove that with positive probability none of the "bad" events hold, it suffices to verify the following inequality:

$$8\frac{d}{x} + \sum_{k=2}^{m} 2^{(k+1)} \frac{d^{(k-1)(1+\beta)}}{x^{k-1}} + 2(m+1)8^{m+1}\frac{d^m}{x^{m-1}} + \sum_{i=1}^{s} 2n_i 4^{n_i} \frac{d^{(n_i-2)(1+\beta)}}{x^{n_i-2}} \leq 1$$

We now substitute $x = Cd^{1+\frac{1}{m-1}}$ where $C = 64(m+1)^3 s$. Using the facts that $\beta = \frac{1}{m-1}$ and $n_i \leq m$ for $i \in [1, s]$, we see that it suffices to verify:

$$\frac{1}{8m^3s} + \frac{1}{32ms} + \frac{2(m+1)8^{m+1}}{(4m+4)^{3m-3}s} + \frac{1}{4m^2} \leq 1$$

The above inequality can easily be seen to be true for any $m \geq 2$, $s \geq 1$.

Thus by Lemma 1, with positive probability, none of the bad events occurs and hence there exists a $(2, \mathcal{F})$-coloring using $O(d^{1+\frac{1}{m-1}})$ colors. This completes the proof of Theorem 1.

3 Relating $\chi_{j,\mathcal{F}}(G)$ and $\chi_o(G)$

We now prove Theorem 2 which relates oriented chromatic number and the forbidden subgraph colorings.

Proof of Theorem 2. Let $G = (V, E)$ be an undirected graph such that $G \notin Forb(\mathcal{F})$ and let $\overrightarrow{G} = (V, A)$ be an arbitrary orientation of $E(G)$. Since $G \notin Forb(\mathcal{F})$, we have $k \geq \chi_{j,\mathcal{F}} \geq j + 1$. Let $V_1, ..., V_k$ be the color classes of V with respect to a $(j, \mathcal{F})$-subgraph coloring c of $V(G)$ using k colors. Let $\mathcal{T}$ be the collection of subsets obtained by partitioning $[1, k]$ into at most $\lceil \frac{k}{\lfloor j/2 \rfloor} \rceil$ subsets of size at most $\lfloor j/2 \rfloor$ each. Note that $|\mathcal{T}|$ is at most $\lceil \frac{2k}{j} \rceil$ if j is even and is at most $\lceil \frac{2k}{j-1} \rceil$ if j is odd. Let $\mathcal{S}$ be the collection defined by

$$\mathcal{S} = \{T \cup T' : T, T' \in \mathcal{T},\ T \neq T'\}.$$

It follows that

(i) Each $S \in \mathcal{S}$ is a set of size at most j.
(ii) for every $l, m \in [1, k]$, there exists a $S \in \mathcal{S}$ with $l, m \in S$,
(iii) for each $i \in [k]$, i is a member of at most $\lceil \frac{k}{\lfloor j/2 \rfloor} \rceil - 1$ sets in $\mathcal{S}$. Let $\mathcal{S}_i$ be defined by $\mathcal{S}_i = \{S \in \mathcal{S} : i \in S\}$.

For each $S \in \mathcal{S}$, let $\overrightarrow{G}_S$ denote the induced subgraph $\overrightarrow{G}[\cup_{i \in S} V_i]$. Clearly $G_S \in Forb(\mathcal{F})$, since $(V_1, \ldots, V_k)$ is a $(j, \mathcal{F})$-subgraph coloring.

Let c_S be an oriented coloring of $\overrightarrow{G}_S$ using at most t colors.

Assume an ordering $\{S_1, S_2, \ldots\}$ on the members of $\mathcal{S}$. We now define a new coloring ϕ of $V(G)$: Fix any i and let $\mathcal{S}_i = \{S_{i_1}, \ldots, S_{i_l}\}$ be the members of $\mathcal{S}_i$ where we have $l \leq \lceil \frac{k}{\lfloor j/2 \rfloor} \rceil - 1$. For each $v \in V_i$,

$$\phi(v) = (c(v), c_{S_{i_1}}(v), ..., c_{S_{i_l}}(v)).$$

Clearly, ϕ is a proper coloring of $V(\overrightarrow{G})$ because of the component c. We now prove that it is an oriented coloring. If it is not an oriented coloring, then there are four vertices x, y, z, t of $\overrightarrow{G}$ such that $(x, y) \in A$ and $(z, t) \in A$ with $\phi(x) = \phi(t)$ and $\phi(y) = \phi(z)$. By the definition of ϕ, x and t (respectively y and z) belong to the same V_i (respectively V_j) where $i = c(x) = c(t)$ and $j = c(y) = c(z)$. Let S be any set in $\mathcal{S}$ containing i and j where $S \in \mathcal{S}_i \cap \mathcal{S}_j$ and $x, y, z, t \in V(\overrightarrow{G}_s)$. By the definition of ϕ, we have $c_S(x) = c_S(t)$ and $c_S(y) = c_S(z)$. But this contradicts the fact that c_S is an oriented coloring of $\overrightarrow{G}_S$.

The number of possible values of $\phi(v)$ is at most $kt^{\lceil \frac{k}{\lfloor j/2 \rfloor} \rceil - 1}$. This number is $kt^{\lceil \frac{2k-j}{j} \rceil}$ if j is even and is $kt^{\lceil \frac{2k-j+1}{j-1} \rceil}$ if j is odd. This proves Theorem 2.

4 $(2, \mathcal{F})$-Subgraph Colorings of Graphs on Surfaces

By applying the bound of Theorem 1 which holds for general graphs, we obtain a bound on $\chi_{2,\mathcal{F}}(G)$ for graphs embeddable on surfaces, provided the members of $\mathcal{F}$ have minimumdegree at least 2. This bound was stated in Theorem 4 and is proved in this section.

The proof is essentially the proof of [2] extended to a more general setting. Hence, we only provide the sketch of the proof.

4.1 Proof of Theorem 4

We follow the proof of [2]. Assume the theorem is false for a surface S with Euler characteristic $-\gamma \leq 0$, and let G be a graph embeddable on it, with a minimum number of vertices, which is a minimal counterexample to the theorem. Let H be G with (possibly multiple) edges added to triangulate S. Clearly $deg_G(v) \leq deg_H(v)$ for all vertices v of G. Suppose $V(G) = V(H) = \{v_1, ..., v_n\}$, where $deg_H(v_1) \leq deg_H(v_2) \leq ... \leq deg_H(v_n)$. If $\gamma = 0$, define $h_1 = 0$ and $h_2 = 0$. Otherwise, define $h_1 := \lceil c\gamma^{\frac{m}{2m-1}} \rceil$ and $h_2 := \lfloor 6\gamma/h_1 \rfloor$ $(\leq 6\gamma^{\frac{m-1}{2m-1}}/c)$, where c is an absolute constant, to be chosen later. Let $d := deg(v_{n-h_1})$. The proof will split on the size of d.

Case I: $d \leq (4/3)h_2 + 9$. In this case, the induced subgraph of G on $\{v_1, ..., v_n\}$ has maximum degree at most d, and thus has a $(2, \mathcal{F})$-subgraph coloring using at most $\lceil Cd^{m/(m-1)} \rceil$ colors, by Theorem 1. Coloring the remaining vertices of G with h_1 new colors that have not been used before gives a $(2, \mathcal{F})$-subgraph coloring of G with at most

$$\lceil C((4/3)h_2 + 9)^{m/(m-1)} \rceil + h_1 \leq C(8\gamma^{\frac{(m-1)}{(2m-1)}}/c + 9)^{m/(m-1)} + c\gamma^{m/(2m-1)} + 2$$

colors. An appropriate choice of constant values (independent of γ) for A, B and c shows that this is smaller than $A\gamma^{m/(2m-1)} + B$, implying that in this case G cannot be a counterexample.

Case II: $d \geq (4/3)h_2 + (28/3)$. We charge each vertex as follows. Define $charge'(v_i) = 6 - deg_H(v_i)$ for $1 \leq i \leq n - h_1$, and $charge'(v_i) = -deg_H(v_i)/4$ for $n - h_1 + 1 \leq i \leq n$.

In this case, using the discharging method and an inductive argument explained in [2], it can be shown that there exists a vertex v such that (i) $G - v$ is $(2, \mathcal{F})$-subgraph colorable using $A\gamma^{m/(2m-1)} + B$ colors, (ii) v can be properly colored with a color i so that for any other color j, any connected component containing v in the union of color classes i and j must have a vertex of degree one. Thus a $(2,\mathcal{F})$-subgraph coloring of $G-v$ can be extended to a $(2,\mathcal{F})$-subgraph coloring of G using no additional color. This is a contradiction to our assumption that G is a minimum counter example. This completes the proof.

4.2 Proof of Theorem 5

The proof is based on an approach similar to the one used in [2]. It uses the following lemma whose proof follows from the proof of a lower bound presented in [1].

Lemma 2. *Let $\mathcal{F}$ and m be as described in Theorem 5. Let $G = G(n,p)$ be the random graph on $\{1,\ldots,n\}$ where each potential edge is chosen independently with probability $p = c\left(\frac{\log n}{n}\right)^{1/m}$ for suitable positive constant c which depends only on $\mathcal{F}$. Then, almost surely, G is connected and has at most $cn^{\frac{(2m-1)}{m}}(\log n)^{\frac{1}{m}}$ edges and satisfies $\chi_{2,\mathcal{F}}(G) = \Omega(n)$.*

Let G be a connected graph on at most $O(n^{(2m-1)/m}(\log n)^{1/m})$ edges and satisfying $\chi_{2,\mathcal{F}}(G) = \Omega(n)$ guaranteed by Lemma 2. Let G be embedded on a surface of characteristic $-\gamma$ for the smallest $\gamma \geq 0$ possible. Let $e = |E(G)|$. By an application of Euler's formula, one can show (as shown in [2]) that $\gamma > n^{(2m-1)/m}$, and hence $\log\gamma > (2m-1)(\log n)/m$ and also that $\gamma = O\left(n^{\frac{(2m-1)}{m}}(\log\gamma)^{\frac{1}{m}}\right)$. Hence, $\chi_{2,\mathcal{F}}(G) = \Omega(n) = \Omega\left(\gamma^{\frac{m}{(2m-1)}}/(\log\gamma)^{\frac{1}{(2m-1)}}\right)$.

5 An Improved Bound on the Oriented Chromatic Number

In [3], Kostochka, Sopena and Zhu showed that the oriented chromatic number of any graph G of maximum degree k is at most $2k^2 2^k$. They prove this result using probabilistic arguments which can in fact be refined so that we obtain the following improvement of this result.

Theorem 7. *If G is any graph of maximum degree k and degeneracy d, then its oriented chromatic number $\chi_o(G)$ is at most $16kd2^d$.*

This replaces a factor $k2^k$ by $d2^d$ and will result in a better bound for those G having $d \ll k$.

As in [3], we prove (using probabilistic arguments) the following lemma. Before that, we recall the following notation from [3]. For an oriented graph $G = (V,A)$ and a subset $I = \{x_1,\ldots,x_i\}$ of V and a vertex $v \in V \setminus I$ such that v is adjacent to each x_j, we use $F(I,v,G)$ to denote the vector $a = (a_1,\ldots,a_i)$ where, for each $j \leq i$, $a_j = 1$ if $(x_j,v) \in A$ and $a_j = -1$ if $(v,x_j) \in A$.

Lemma 3. *Let d, k be positive integers with $d \leq k$ and $k \geq 5$. There exists a tournament $T = (V, A)$ on $t = 16kd2^d$ vertices with the following property:*

For each $i, 0 \leq i \leq d$, for each $I \subseteq V$, $|I| = i$, and for each $a \in \{1, -1\}^i$, there exist at least $kd + 1$ vertices $v \in V \setminus I$ with $F(I, v, T) = a$.

We now give the proof of Theorem 7 assuming Lemma 3 whose proof is omitted due to lack of space.

Proof of Theorem 7. Let $G = (V, E)$ be any graph of maximum degree k and degeneracy d. If $d \leq 1$, then G is a forest and hence its $\chi_o(G) \leq 3$ as shown in [6]. For $d \geq 2$ and $k \leq 4$, the result follows from a bound of $(2k - 1)2^{2k-2}$ derived in [6]. Hence, we assume that $k \geq 5$ and $d \geq 2$. Consider a linear ordering $(v_n, \ldots, v_1)$ of V such that for each $i \leq n$, v_i has at most d neighbors in the subgraph G_i induced by $V_i = \{v_1, \ldots, v_i\}$. Let T be the tournament on $t = 16kd2^d$ vertices specified in Lemma 3. Let G' be any orientation of G. We inductively color vertices of G' in the order $(1, \ldots, n)$ in such a way that after the coloration of the first m vertices:

(1) The partial coloring $f(v_1), \ldots, f(v_m)$ is a valid oriented coloring of G'_m using vertices of T;
(2) For each v_j with $j > m$, all neighbors of v_j in V_m are colored with distinct colors.

Now, we need to color v_{m+1} so that (1) and (2) hold for $f(v_{m+1})$ as well. For this, let $\{y_1, \ldots, y_i\} \subseteq V_m$ be the neighbors of v_{m+1} in V_m each colored with distinct colors (because of (2)) from $I = \{f(y_1), \ldots, f(y_i)\}$. Note that $i \leq d$. Let $a = F(\{y_1, \ldots, y_i\}, v_{m+1}, G'_{m+1})$. Let $K = \{w \in V(T) \setminus I : F(I, w, T) = a\}$. By Lemma 3, we know that $|K| \geq kd + 1$. Now, there can be at most kd paths of the form (v_{m+1}, u, v_j) such that $u \in V \setminus V_{m+1}$ is a neighbor of v_{m+1} in G and $v_j, j \leq m$ is a neighbor of u in V_m. Let $B \subseteq V_m$ be the set of all such v_j's and let $f(B)$ be the set of their colors with $|f(B)| \leq kd$. Now, color v_{m+1} with any color from $K \setminus f(B)$ and one can easily check that $f(v_{m+1})$ satisfies both (1) and (2), thus extending the coloring inductively. This proves Theorem 7.

6 Conclusions and Open Problems

We showed a relation between forbidden subgraph colorings and oriented colorings. In particular, we obtained an upper bound for oriented chromatic number in terms of low treewidth colorings and found an upper bound of $O(2^{g^{1/2+o(1)}})$ for the oriented chromatic number of graphs of genus g. However, we believe that this bound is not tight. In fact, we believe in the following

Conjecture: There exist absolute positive constants c_1, c_2 such that : if G is a graph of genus at most g, then $\chi_o(G) \leq c_1 2^{c_2\sqrt{g}}$.

Further, it would be interesting to obtain bounds for the (j, k)-treewidth chromatic number (for graphs of bounded genus), when $j > 2$. We also pose the following interesting and challenging open problem.

Open Problem: Determine if there is a k such that $\chi^{tw}_{2,k}(G) \leq 4$ for all planar graphs G and find the smallest such k if it exists.

Note that if we replace 4 by 5 in the above inequality, then the answer is yes for $k = 1$ since it has been shown by Borodin [11] that $a(G) \leq 5$ for any planar graph G. Also, this bound is tight as Grünbaum [4] has obtained an infinite family of planar graphs having no acyclic 4-coloring.

References

1. Aravind, N.R., Subramanian, C.R.: Bounds on proper colorings with restrictions on the union of color classes. Submitted to a Journal
2. Alon, N., Mohar, B., Sanders, D.P.: On acyclic colorings of graphs on surfaces. Israel Journal of Mathematics 94, 273–283 (1996)
3. Kostochka, A.V., Sopena, E., Zhu, X.: Acyclic and oriented chromatic numbers of graphs. Journal of Graph Theory 24(4), 331–340 (1997)
4. Grünbaum, B.: Acyclic colorings of planar graphs. Israel Journal of Mathematics 14, 390–408 (1973)
5. Albertson, M.O., Berman, D.M.: The acyclic chromatic number. Congr. Numer. 17, 51–60 (1976)
6. Sopena, E.: The chromatic number of oriented graphs. Journal of Graph Theory 25(2), 191–205 (1997)
7. Raspaud, A., Sopena, E.: Good and semi-strong colorings of oriented planar graphs. Inf. Process. Lett. 51(4), 171–174 (1994)
8. Nesetril, J., de Mendez, P.O.: Tree-depth, subgraph coloring and homomorphism bounds. Eur. J. Comb. 27(6), 1022–1041 (2006)
9. Alon, N., McDiarmid, C., Reed, B.: Acyclic coloring of graphs. Random Struct. Algorithms 2(3), 277–288 (1991)
10. Ringel, G., Youngs, J.W.T.: Solution of the heawood map coloring problem. Proc. Nat. Acd. Sci. U.S.A. 60, 438–445 (1968)
11. Borodin, O.V.: Acyclic colorings of planar graphs. Discrete Mathematics 25(3), 211–236 (1979)